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Worldsheet instantons and torsion curves. Part A: direct computation

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Abstract: As a first step towards studying vector bundle moduli in realistic heterotic compactifications, we identify all holomorphic rational curves in a Calabi-Yau threefold X with $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ Wilson lines. Computing the homology, we find that $H_2(X, \mathbb{Z}) = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. The *torsion* curves complicate our analysis, and we develop techniques to distinguish the torsion part of curve classes and to deal with the non-toric threefold X. In this paper, we use direct A-model computations to find the instanton numbers in each integral homology class, including torsion. One interesting result is that there are homology classes that contain only a single instanton, ensuring that there cannot be any unwanted cancellation in the non-perturbative superpotential.

Keywords: Topological Strings, Superstrings and Heterotic Strings, Solitons Monopoles and Instantons.

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Contents

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1. Introduction

The goal of this paper is to count world sheet instantons on a certain Calabi-Yau threefold X. Now that in itself was essentially solved by mirror symmetry a long time ago [1], but here there is an important subtlety that does not appear in the most simple Calabi-Yau constructions. This subtlety is the appearance of *torsion* curve classes in the degree-2 homology of X. In particular,[1]

$$H_2\big(X,\mathbb{Z}\big) = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3, \tag{1.1}$$

which contains the *torsion*[2] *subgroup* $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. There are already a few known examples of such Calabi-Yau manifolds with torsion curves [2–6], but the proper instanton counting has never been done before.

Still, the question remains: Why should we be interested in this? We are really interested in instanton corrections to the heterotic MSSM constructed in [7–9], in particular to the superpotential for bundle moduli. Classically, there is no superpotential generated for the vector bundle moduli if the bundle is at a smooth point in its moduli space (see also [10] for a non-smooth example). If there were no potential generated for the vector bundle moduli then there would be no hope of stabilizing all moduli, a phenomenological disaster. As is well known, only genus 0 instantons (rational curves) contribute to the superpotential, and we will exclusively consider these in the following. The general hope is that the E_8 gauge bundle will give rise to instanton corrections generating a non-vanishing

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[1] In the following, $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ always denotes the integers mod 3. Similarly, we write $(\mathbb{Z}_3)^n = \oplus_n \mathbb{Z}_3 = \mathbb{Z}_3 \oplus \cdots \oplus \mathbb{Z}_3$ for the Abelian group generated by n generators of order 3.

[2] Not to be confused with the completely unrelated torsion tensor of a connection.

superpotential which is sufficiently complicated to stabilize moduli [11 – 17]. However, this is far from obvious, especially in view of unexpected cancellations between instantons in the same homology class found in [18 – 22]. Now in our case [23 – 26] the Calabi-Yau threefold is not a toric complete intersection and the vector bundle does not come from the ambient space, so the above arguments do not apply. Still, it is not, a priori, clear that the instanton contributions do not cancel for some other reason. However, as we are going to show in the following, the simplest smooth rigid rational curves in X are alone in their homology class, and no such cancellation can occur. In fact, they contribute to the vector bundle superpotential as will be explained elsewhere.

Another independent motivation is the following. Any (real 2-dimensional) surface in a torsion homology class cannot be contracted by definition. Yet integrating any closed 2-form over this surface must give zero, since a multiple of the surface is contractible. So whatever minimal volume surface there is in a torsion homology class, its volume is not the integral over the Kähler form. In particular, the curve cannot be holomorphic and a D-brane carrying the corresponding K-theory[3] charge cannot preserve any supersymmetry (assuming no background fluxes).

As a final motivation, we note that, on general grounds, $H_2(X,\mathbb{Z})_{\text{tors}} = H^3(X,\mathbb{Z})_{\text{tors}}$. Hence, if there is torsion then there is a possibility for fractional Chern-Simons invariants. It was argued in [28] that under favorable circumstances this can generate a potential for complex structure moduli, Kähler moduli, and dilaton.

Given these motivations, we will only complete the first step and count rational curves on X. Really, this means finding the instanton correction $\mathcal{F}^{\text{np}}_{X,0}$ to the prepotential of the topological string. This is usually written as a (convergent) power series in h^{11} variables $q_a = e^{2\pi \mathrm{i} t^a}$. The novel feature of the 3-torsion curves on X is that for each 3-torsion generator we need an additional variable b_j such that $b_j^3 = 1$. The Fourier series of the prepotential on X becomes

$$\mathcal{F}^{\text{np}}_{X,0}(q_1, q_2, q_3,\, b_1, b_2) = \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z} \\ m_1,m_2 \in \mathbb{Z}_3}} n_{(n_1,n_2,n_3,m_1,m_2)} \operatorname{Li}_3\left(q_1^{n_1} q_2^{n_2} q_3^{n_3} b_1^{m_1} b_2^{m_2}\right), \tag{1.2}$$

where $N_{(n_1,n_2,n_3,m_1,m_2)}$ is the instanton number in the curve class $(n_1, n_2, n_3, m_1, m_2)$. Realizing this, we will investigate a number of complementary ways to determine this prepotential:

- Part of the prepotential of the universal cover $\widetilde{X}$ was computed directly in [29], and by carefully descending to the quotient $X = \widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ we can compute the corresponding part of the prepotential of X.

- The same part of the prepotential of X can also be computed by directly counting curves on X.

These two A-model calculations will be carried out in this paper, which we therefore entitle Part A. By construction, these computations only yield a part of the prepotential,

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[3] We remind the reader that on a Calabi-Yau threefold $H^{\text{ev}}(Y,\mathbb{Z}) \simeq K^0(Y)$ and $H^{\text{odd}}(Y,\mathbb{Z}) \simeq K^1(Y)$, so in particular the torsion parts are identical [27].

although an important one. To overcome this limitation, we will use the B-model and mirror symmetry in Part B, the companion paper [30]. More precisely, we will do the following:

- Mirror symmetry for the toric complete intersection $\widetilde{X}$ provides an algorithm to compute instanton numbers. Unfortunately, there are many non-toric divisors which cannot be treated this way. It turns out that, after descending to X, precisely the torsion information is lost. In this approach, one can only compute $\mathcal{F}^{\text{np}}_{X,0}(q_1, q_2, q_3,\, 1, 1)$.
- As a pleasant surprise we find strong evidence that the manifold X of principal interest is self-mirror. In particular, we attempt to compute the instanton numbers on the mirror X^* by descending from the covering space $\widetilde{X}^*$. The toric embedding of $\widetilde{X}^*$ is such that all 19 divisors are toric. A complete analysis including the full $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ torsion information would be feasible after some straightforward efficiency improvement of existing software [31].
- Although the full quotient $X = \widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is not toric, it turns out that a certain partial quotient $\widetilde{X}/\mathbb{Z}_3$ can be realized as a toric variety. That way, one only has to deal with $h^{11}(\widetilde{X}/\mathbb{Z}_3) = 7$ parameters, which is manageable on a computer. On the mirror $(\widetilde{X}/\mathbb{Z}_3)^*$, all divisors are toric and we can compute the expansion $\mathcal{F}^{\text{np}}_{X,0}(q_1, q_2, q_3,\, 1, b_2)$ to any desired degree. A symmetry argument allows one to recover the b_1 dependence as well.

The result of these calculations is the complete prepotential $\mathcal{F}^{\text{np}}_{X,0}(q_1, q_2, q_3,\, b_1, b_2)$. The instanton numbers can be numerically computed for any integral homology class, limited only by computing power. We preview these results in the conclusion of this paper. A complete discussion is presented in [30].

To prepare the ground, we first have to compute the torsion curves on X. We will do this in I of the present paper. In sections 2 and 3 we define the manifold $X = \tilde{X}/G$ as a free quotient and introduce appropriate bases for the homology and cohomology of the cover. In 4 we compute the group homology and cohomology of $\mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ with coefficients in the appropriate (co)homology groups. These results are used in 5 to compute the integral homology groups of the full and of the partial quotient with appropriate spectral sequences. 5.1 contains a non-technical summary of the torsion curves.

In II of the present paper, we proceed to do the A-model analysis of the instanton numbers. As a simpler example without torsion curves, we first recapitulate certain free quotients of the quintic threefold in 6. Subsequently, in sections 7 and 8 we investigate X using the aforementioned A-model techniques. Finally, we present our conclusions in 9. An easily readable overview over these results can be found in [32].

Part I

Torsion curves

2. The Calabi-Yau threefold

2.1 Covering space

The Calabi-Yau manifold X we are going to investigate is constructed as a free $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ quotient of its universal covering space $\widetilde{X}$. As usual, instead of working with a non-simply connected manifold it is technically easier to analyze the group action on its covering space. The simply connected Calabi-Yau threefold $\widetilde{X}$ is one of Schoen's threefolds [33]. It can be described in various ways, including the fiber product of two dP_9 surfaces, resolution of a certain T^6 orbifold [34], or a complete intersection. For concreteness we adopt the latter viewpoint in this section. One first introduces the ambient variety $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ with homogeneous coordinates

$$\Big([x_0 : x_1 : x_2],\ [t_0 : t_1],\ [y_0 : y_1 : y_2]\Big) \in \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2\,. \tag{2.1}$$

A generic complete intersection of a degree $(0, 1, 3)$ and a degree $(3, 1, 0)$ polynomial is a smooth Calabi-Yau threefold, but does not admit a non-trivial $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action. However, the polynomials

$$t_0\Big(x_0^3 + x_1^3 + x_2^3\Big) + t_1\Big(x_0x_1x_2\Big) = F_1 \tag{2.2a}$$

$$\big(\lambda_1 t_0 + t_1\big)\Big(y_0^3 + y_1^3 + y_2^3\Big) + \big(\lambda_2 t_0 + \lambda_3 t_1\big)\Big(y_0y_1y_2\Big) = F_2, \tag{2.2b}$$

where λ_1, λ_2, λ_3 are complex parameters, are invariant under the $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ action generated by ($\zeta = e^{\frac{2\pi i}{3}}$)

$$g_1 : \begin{cases} [x_0 : x_1 : x_2] \mapsto [x_0 : \zeta x_1 : \zeta^2 x_2] \\ [t_0 : t_1] \mapsto [t_0 : t_1] \text{ (no action)} \\ [y_0 : y_1 : y_2] \mapsto [y_0 : \zeta y_1 : \zeta^2 y_2] \end{cases} \tag{2.3a}$$

and

$$g_2 : \begin{cases} [x_0 : x_1 : x_2] \mapsto [x_1 : x_2 : x_0] \\ [t_0 : t_1] \mapsto [t_0 : t_1] \text{ (no action)} \\ [y_0 : y_1 : y_2] \mapsto [y_1 : y_2 : y_0] \end{cases} \tag{2.3b}$$

This group action has fixed points in the ambient variety $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$, but these do not satisfy eqs. (2.2a) and (2.2b). Hence, this $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action on the complete intersection Calabi-Yau threefold

$$\widetilde{X} \;=\; \Big\{\big([x_0 : x_1 : x_2], [t_0 : t_1], [y_0 : y_1 : y_2]\big)\Big| F_1 = 0,\, F_2 = 0\Big\} \;\subset\; \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2 \tag{2.4}$$

is free.

JHEP10(2007)022

We point out that this $\mathbb{Z}_3 \times \mathbb{Z}_3$ action is slightly different from the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action investigated within the context of an heterotic standard model [35]. The group action we discuss in this paper "does not act on the base $\mathbb{P}^1$" and, hence, is not included in the classification [35]. The reason we are using the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action defined above is that it is more amenable to toric methods, which will be important for the B-model computation later in this paper. However, the curve counting can be easily extended to the MSSM manifold [35], which we will present elsewhere.

Finally, let us review some facts about the homology and cohomology of the universal cover $\widetilde{X}$, see [33, 36]. The Hodge diamond of the Calabi-Yau threefold $\widetilde{X}$ is self-mirror and given by

$$h^{p,q}\big(\widetilde{X}\big) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 19 & & 0 & \\ 1 & & 19 & & 19 & & 1 \\ & 0 & & 19 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \Rightarrow \qquad H^i_{\text{de Rham}}\big(\widetilde{X}, \mathbb{R}\big) = \begin{cases} \mathbb{R} & i = 6 \\ 0 & i = 5 \\ \mathbb{R}^{19} & i = 4 \\ \mathbb{R}^{40} & i = 3 \\ \mathbb{R}^{19} & i = 2 \\ 0 & i = 1 \\ \mathbb{R} & i = 0. \end{cases} \tag{2.5}$$

In general the dimension of the i-th de Rham cohomology is the same as the rank of the i-th integral cohomology group, but the latter might also contain torsion information which is not captured by de Rham cohomology. However, a smooth complete intersection in a smooth toric variety does not have any torsion in its integral cohomology [37]. This determines the integral cohomology, and Poincaré duality eq. (A.2) then yields the integral homology groups. We conclude that

$$H_{6-i}\big(\widetilde{X}, \mathbb{Z}\big) = H^i\big(\widetilde{X}, \mathbb{Z}\big) = \begin{cases} \mathbb{Z} & i = 6 \\ 0 & i = 5 \\ \mathbb{Z}^{19} & i = 4 \\ \mathbb{Z}^{40} & i = 3 \\ \mathbb{Z}^{19} & i = 2 \\ 0 & i = 1 \\ \mathbb{Z} & i = 0. \end{cases} \tag{2.6}$$

2.2 The quotient

Having constructed $\widetilde{X}$ with a free $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action, we define

$$X = \widetilde{X} \Big/ \big(\mathbb{Z}_3 \times \mathbb{Z}_3\big). \tag{2.7}$$

On general grounds, X is again a smooth Calabi-Yau threefold with fundamental group $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$. Since the defining equations (2.2a), (2.2b) allow for three independent coefficients up to $PGL(3) \times PGL(2) \times PGL(3)$ coordinate changes if one wants to preserve the $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry, we expect that there are $h^{21}(X) = 3$ complex structure parameters. This turns out to be true, as will be shown in more detail in 4.2.

Moreover, we know the Euler numbers[4] vanish,

$$\chi\big(\widetilde{X}\big) = 2h^{11}\big(\widetilde{X}\big) - 2h^{21}\big(\widetilde{X}\big) = 0 = 9\chi\big(X\big). \tag{2.8}$$

This fixes the Hodge numbers of the quotient $X = \widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ to be

$$h^{p,q}\big(X\big) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 1 & & 3 & & 3 & & 1 \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.9}$$

However, knowing the Betti numbers does not tell us everything about the homology classes of curves. The integral homology groups potentially contain *torsion*, that is, a finite subgroup. For example, as we will show in 5

$$H_2\big(X, \mathbb{R}\big) = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} = \mathbb{R}^3, \quad H_2\big(X, \mathbb{Z}\big) = \mathbb{Z}^3 \oplus \Big(\mathbb{Z}_3 \oplus \mathbb{Z}_3\Big). \tag{2.10}$$

The subgroup $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ consisting of 9 elements is such a torsion subgroup. Clearly, explicit knowledge of *all* curve homology classes is important when counting curves on X.

3. Group action

3.1 Projections

As usual, instead of analyzing the quotient $X = \widetilde{X}/G$ directly we will look at the $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ action on the covering space. In this section, we find it particularly useful to exploit the property that $\widetilde{X}$ has two projections to dP_9 surfaces. To see this, note that a degree $(3, 1)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$ is such a dP_9 surface, also called a rational elliptic surface. Moreover, the defining equations (2.2a) and (2.2b) do not depend on $[y_0 : y_1 : y_2]$ and $[x_0 : x_1 : x_2]$, respectively. Hence, eq. (2.2a) and eq. (2.2b) define dP_9 surfaces with natural projections $\pi_1 : \widetilde{X} \to B_1$, $\pi_2 : \widetilde{X} \to B_2$. Finally, each B_1, B_2 projects to the common $\mathbb{P}^1$, yielding a commutative diagram

$$\begin{tikzcd}
\dim_{\mathbb{C}} = 3 : & & \widetilde{X} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\dim_{\mathbb{C}} = 2 : & B_1 \arrow[dr] & & B_2 \arrow[dl] \\
\dim_{\mathbb{C}} = 1 : & & \mathbb{P}^1 . &
\end{tikzcd} \tag{3.1}$$

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[4] Note that $\widetilde{X}$ will turn out to be self-mirror. Nevertheless, instanton corrections are present, part of which were been computed in [29, 38, 39]. There is a common misconception based on the free $K3 \times T^2/\mathbb{Z}_2$ orbifold investigated in [5, 6] that self-mirror threefolds do not receive quantum corrections to the classical moduli space. Indeed, in that case, all rational curves come in families which happen not to contribute [40], that is, their Gromov-Witten invariants vanish. However, this is not due to $K3 \times T^2/\mathbb{Z}_2$ being self-mirror.

By definition, this means that $\widetilde{X}$ is the fiber product of two dP_9 surfaces, that is, $\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2$. In other words, $\widetilde{X}$ is elliptically fibered over each B_i, $i = 1, 2$, and each B_i is again elliptically fibered over the same $\mathbb{P}^1$. In the remainder of this section, we are going to detail the properties of these dP_9 surfaces.

The $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action descends to B_1, B_2. Moreover, since the action is trivial on the $\mathbb{P}^1$, it must be the translation[5] by two independent sections. The existence of two sections of order three determines the Kodaira fibers and Mordell-Weil group uniquely [41] to be

$$\begin{aligned} \mathrm{Sing}(B_1) &= \mathrm{Sing}(B_2) = 4I_3, \\ MW(B_1) &= MW(B_2) = \mathbb{Z}_3 \oplus \mathbb{Z}_3. \end{aligned} \tag{3.2}$$

Recall that the Mordell-Weil group is the set of all sections (which depends on the moduli of the dP_9 surface) together with a group law "$\boxplus$". The Mordell-Weil sum[6] $\alpha \boxplus \beta$ of two sections α, β is the fiberwise sum. In other words,

$$\alpha \boxplus \beta = t_\alpha(\beta) = t_\beta(\alpha). \tag{3.3}$$

Let us label[7] the generating sections μ and ν on B_i, $i = 1, 2$ such that

$$MW(B_i) = \Big\{\sigma,\, \mu,\, \mu \boxplus \mu,\, \nu,\, \nu \boxplus \mu,\, \nu \boxplus \mu \boxplus \mu,\, \nu \boxplus \nu,\, \nu \boxplus \nu \boxplus \mu,\, \nu \boxplus \nu \boxplus \mu \boxplus \mu\Big\}, \tag{3.4}$$

with σ being the zero section. Furthermore, note that each vertical I_3 fiber is composed of three irreducible components, intersecting in a triangle. We denote the i-th component of the j-th I_3 Kodaira fiber by θ_{ji}. Up to re-indexing the divisors, there is only one possible intersection pattern between the horizontal and vertical divisors, namely

$(-)\cdot(-)$	θ_{10}	θ_{11}	θ_{12}	θ_{20}	θ_{21}	θ_{22}	θ_{30}	θ_{31}	θ_{32}	θ_{40}	θ_{41}	θ_{42}
σ	1	0	0	1	0	0	1	0	0	1	0	0
μ	1	0	0	0	1	0	0	1	0	0	1	0
ν	0	1	0	1	0	0	0	0	1	0	1	0

(3.5)

Finally, denote the class of an elliptic fiber by f.

Recall the Hodge diamond, homology, and cohomology of dP_9 surfaces,

$$h^{p,q}(B_i) = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}, \qquad H_{4-i}(B_i, \mathbb{Z}) = H^i(B_i, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 4 \\ 0 & i = 3 \\ \mathbb{Z}^{10} & i = 2 \\ 0 & i = 1 \\ \mathbb{Z} & i = 0. \end{cases} \tag{3.6}$$

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[5] A point z_0 on an elliptic surface $\mathbb{C}/\Lambda$ defines a group action $z \mapsto z + z_0$. A section of the elliptic fibration B_i consists of a point in each fiber. Hence, a section s defines a group action $t_s : B_i \to B_i$ by translation along each fiber.

[6] We point out that the Mordell-Weil sum "$\boxplus$" must be distinguished from the sum of homology classes, which we write as "+". For example, $\alpha \boxplus \beta$ is again a section whereas $\alpha + \beta$ is a two-section.

[7] In the following, it will always be clear from the context whether we are referring to B_1 or B_2. Hence we use the same symbol for divisors in B_1 and B_2.

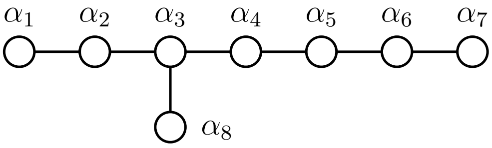

Figure 1: The E_8 Dynkin diagram.

Therefore, although the above $9 + 3 \cdot 4 + 1$ divisors generate $H_2(B_i, \mathbb{Z}) = \mathbb{Z}^{10}$, they cannot be linearly independent. It is a straightforward task to identify all relations, which we will do in B. One possible integral basis [42, 43] is

$$H_2(B_i, \mathbb{Z}) = \operatorname{span}_{\mathbb{Z}} \Big\{\sigma,\, f,\, \theta_{11},\, \theta_{21},\, \theta_{31},\, \theta_{32},\, \theta_{41},\, \theta_{42},\, \mu,\, \nu\Big\}, \tag{3.7}$$

and we will use this integral basis in the following.

3.2 The $\mathbf{E_8}$ lattice

There is another special basis for the homology of the dP_9 surfaces in addition to eq. (3.7). This other basis is the natural basis choice for a generic dP_9 surface B, that is, one with $12I_0$ singular fibers. In that case the Mordell-Weil group is E_8. This means that the quotient

$$H_2\big(B, \mathbb{Z}\big)\Big/ \operatorname{span}_{\mathbb{Z}} \big\{\sigma, f\big\} = MW(B) = \Lambda_{E_8} \tag{3.8}$$

is the E_8 root lattice with respect to the height pairing

$$\langle s_1, s_2\rangle = 1 + s_1 \cdot \sigma + s_2 \cdot \sigma - s_1 \cdot s_2. \tag{3.9}$$

Therefore, one obvious integral basis choice is to pick 8 simple roots together with σ and f,

$$H_2(B_i, \mathbb{Z}) = \operatorname{span}_{\mathbb{Z}} \Big\{\sigma,\, f,\, \alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4,\, \alpha_5,\, \alpha_6,\, \alpha_7,\, \alpha_8\Big\}. \tag{3.10}$$

Of course, the generic dP_9 does not have the $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action which we are interested in. For example, the Mordell-Weil lattice in our case needs to be $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ instead of Λ_{E_8}. However, the homology groups do not know about the choice of complex structure. Hence, although, in our case, the homology classes α_i cannot be represented by sections, we can still use the same basis for homology. The 240 roots of E_8 are readily identified as

$$\Phi_{E_8} = \Big\{\alpha \in H_2(B, \mathbb{Z}) \;\Big|\; \alpha \cdot f = 1,\, \alpha \cdot \sigma = 0,\, \alpha \cdot \alpha = -1\Big\}. \tag{3.11}$$

The choice of simple roots is not unique. For convenience, we will make the same choice as in [29]:

$$\begin{aligned}
\alpha_1 &= 2\sigma + 2f - \mu, \\
\alpha_2 &= 2\sigma + 2f - \theta_{21} - \theta_{31} - \theta_{41} - \mu, \\
\alpha_3 &= \theta_{21} + \theta_{31} + \theta_{41} + 2\mu - \nu, \\
\alpha_4 &= 2\sigma + 2f - \theta_{31} - \theta_{32} - \theta_{41} - \mu, \\
\alpha_5 &= 2\sigma + 2f - \theta_{21} - \theta_{41} - \theta_{42} - \mu, \\
\alpha_6 &= -\theta_{11} + \theta_{21} + \theta_{31} + \theta_{41} + \theta_{42} + 2\mu - \nu, \\
\alpha_7 &= 2\sigma + 2f - \theta_{31} - \theta_{41} - \theta_{42} - \mu, \\
\alpha_8 &= -2\sigma - 2f + \theta_{11} + \theta_{31} + 2\theta_{32} + 2\theta_{41} + \theta_{42} + 3\nu.
\end{aligned} \tag{3.12}$$

To clarify, on a generic dP_9 surface B the sections α_i can be added by the usual Mordell-Weil sum "$\boxplus$" defined previously. However, the definition of "$\boxplus$" as fiberwise sum of points on a torus depends on having actual sections, and not just the homology classes. However, while on the special dP_9 surfaces B_1, B_2 the homology classes α_i are still well-defined, they need not contain a section anymore. Nevertheless, we can still define the lattice sum

$$\boxplus : \Lambda_{E_8} \times \Lambda_{E_8} \to \Lambda_{E_8} \tag{3.13}$$

on $\Lambda_{E_8} \subset B_1, B_2$ by taking it to the same as for the generic dP_9 surface B.

3.3 Action on the base

We start by analyzing the base dP_9 surfaces B_1, B_2 which, as discussed above, are again elliptically fibered over $\mathbb{P}^1$. The $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ group action[8] is fiberwise translation

$$g_1 = t_\mu, \quad g_2 = t_\nu \tag{3.14}$$

by the two sections μ, ν of order 3 described previously. Obviously, this maps the fiber to itself, $g_1(f) = g_2(f) = f$. On any section, that is, any element of $MW(B_i)$, the group also acts in the obvious way

$$\begin{aligned}
MW(B_i) &= \operatorname{span}_{\boxplus}\{\mu, \nu\} \Big/ \big(\boxplus_3\, \mu = \boxplus_3 \nu = \sigma\big), \\
g_1(s) &= s \boxplus \mu, \quad g_2(s) = s \boxplus \nu.
\end{aligned} \tag{3.15}$$

Finally, the action on each I_3 Kodaira fiber either maps each irreducible component to itself or cyclically permutes the irreducible components, as explained in [35]. From eq. (3.5) we can read off that

D	θ_{10}	θ_{11}	θ_{12}	θ_{20}	θ_{21}	θ_{22}	θ_{30}	θ_{31}	θ_{32}	θ_{40}	θ_{41}	θ_{42}
$g_1(D)$	θ_{10}	θ_{11}	θ_{12}	θ_{21}	θ_{22}	θ_{20}	θ_{31}	θ_{32}	θ_{30}	θ_{41}	θ_{42}	θ_{40}
$g_2(D)$	θ_{11}	θ_{12}	θ_{10}	θ_{20}	θ_{21}	θ_{22}	θ_{32}	θ_{30}	θ_{31}	θ_{41}	θ_{42}	θ_{40}

(3.16)

---

[8] By abuse of notation we use $G = \{\mathrm{id}, g_1, g_1^2, g_2, g_1g_2, g_1^2g_2, g_2^2, g_1g_2^2, g_1^2g_2^2\}$ for the group action on $\widetilde{X}$ and for the induced action on B_1, B_2.

Using the relations from B we can now express the G action on $H_2(B_i,\mathbb{Z})$ as 10×10 matrices in the basis eq. (3.7). One obtains

$$g_1=\begin{pmatrix} 0&0&0&3&0&0&0&0&-1&-1\\ 0&1&0&3&0&1&0&1&-1&-1\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&-2&0&0&0&0&1&0\\ 0&0&0&-2&0&-1&0&0&1&1\\ 0&0&0&-1&1&-1&0&0&0&1\\ 0&0&0&-2&0&0&0&-1&1&1\\ 0&0&0&-1&0&0&1&-1&0&0\\ 1&0&0&-3&0&0&0&0&2&1\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix},\quad g_2=\begin{pmatrix} 0&0&3&0&0&0&0&0&-1&-1\\ 0&1&3&0&1&0&0&1&-1&-1\\ 0&0&-2&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&-1&0&-1&1&0&0&1&0\\ 0&0&-2&0&-1&0&0&0&1&1\\ 0&0&-2&0&0&0&0&-1&1&1\\ 0&0&-1&0&0&0&1&-1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 1&0&-3&0&0&0&0&0&1&2 \end{pmatrix}. \tag{3.17}$$

3.4 Line bundles

Having determined the group action on the base dP_9 surfaces, we now investigate the action on $\widetilde{X}$. First, recall that by a happy coincidence $h^{2,0}(\widetilde{X})=0$ and therefore

$$\begin{aligned} \text{Pic}\left(\widetilde{X}\right) &= \left\{\text{Algebraic line bundles on } \widetilde{X}\right\} \\ &= \left\{\text{Topological line bundles on } \widetilde{X}\right\} = H^2\big(\widetilde{X},\mathbb{Z}\big) = H_4\big(\widetilde{X},\mathbb{Z}\big). \end{aligned} \tag{3.18}$$

In other words,

- Each line bundle has a unique complex structure.
- A line bundle is uniquely determined by its first Chern class.
- Every line bundle $\mathcal{L}$ can be written as $\mathcal{L}=O_{\widetilde{X}}(D)$, and depends only on the homology class of the divisor $D\in H_4\big(\widetilde{X},\mathbb{Z}\big)$.

Note that the identification $H_2=H^4$ does not involve any duality (see A.1), which will be important later on. To lift from B_i, $i=1,2$ to $\widetilde{X}$, we can use

- Pull back of line bundles: $\pi_i^*:\text{Pic}(B_i)\to\text{Pic}(\widetilde{X})$.
- Pull back in cohomology: $\pi_i^*:H^2\big(B_i,\mathbb{Z}\big)\to H^2\big(\widetilde{X},\mathbb{Z}\big)$.
- Preimage of divisors: $\pi_i^{-1}:H_2\big(B_i,\mathbb{Z}\big)\to H_4\big(\widetilde{X},\mathbb{Z}\big)$.

All these notions commute with the identifications eq. (3.18). However, the pull backs of the $\dim H_2(B_1,\mathbb{Z})+\dim H_2(B_2,\mathbb{Z})=20$ line bundles on the bases cannot be independent in $H_4\big(\widetilde{X},\mathbb{Z}\big)\simeq\mathbb{Z}^{19}$. As was shown in [44–46, 36], the line bundles on $\widetilde{X}$ have a particularly nice description, that is, the pullback of the line bundles to $\widetilde{X}$ yields a generating set of 20 line bundles, which must satisfy one relation. This relation is that $\pi_1^{-1}(f)=\pi_2^{-1}(f)$, both being the Abelian surface fiber of the fibration $\widetilde{X}\to\mathbb{P}^1$. Hence,

$$\begin{aligned} H_4\big(\widetilde{X},\mathbb{Z}\big) &= \Big[\pi_1^{-1}H_2\big(B_1,\mathbb{Z}\big)\oplus\pi_2^{-1}H_2\big(B_2,\mathbb{Z}\big)\Big]\Big/\left\langle\pi_1^{-1}(f)=\pi_2^{-1}(f)\right\rangle \\ &= \text{span}_{\mathbb{Z}}\Big\{\pi_1^{-1}(f)=\pi_2^{-1}(f), \\ &\qquad \pi_1^{-1}(\sigma),\, \pi_1^{-1}(\theta_{11}),\, \pi_1^{-1}(\theta_{21}),\, \pi_1^{-1}(\theta_{31}),\, \pi_1^{-1}(\theta_{32}), \\ &\qquad \pi_1^{-1}(\theta_{41}),\, \pi_1^{-1}(\theta_{42}),\, \pi_1^{-1}(\mu),\, \pi_1^{-1}(\nu), \\ &\qquad \pi_2^{-1}(\sigma),\, \pi_2^{-1}(\theta_{11}),\, \pi_2^{-1}(\theta_{21}),\, \pi_2^{-1}(\theta_{31}),\, \pi_2^{-1}(\theta_{32}), \\ &\qquad \pi_2^{-1}(\theta_{41}),\, \pi_2^{-1}(\theta_{42}),\, \pi_2^{-1}(\mu),\, \pi_2^{-1}(\nu)\Big\}\simeq\mathbb{Z}^{19}. \end{aligned} \tag{3.19}$$

Having determined the geometric action on the divisors of the surfaces B_i in 3.3, one can now easily determine the $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ representation on $H_4(\widetilde{X}, \mathbb{Z})$ in terms of 19×19 integer matrices. Other than to note that we use them in the following for some linear algebra computations, it is not particularly enlightening to present the explicit matrices here. We denote this representation as

$$R^\vee = H_4\big(\widetilde{X}, \mathbb{Z}\big). \tag{3.20}$$

3.5 Curves

Abstractly, the previous subsection boils down to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H^2\big(B_1, \mathbb{Z}\big) \oplus H^2\big(B_2, \mathbb{Z}\big) \xrightarrow{\pi_1^* + \pi_2^*} H^2\big(\widetilde{X}, \mathbb{Z}\big) \longrightarrow 0. \tag{3.21}$$

Recall that the fiber product $\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2$ is a hypersurface in $B_1 \times B_2$. The Poincaré dual (see A.1) sequence

$$0 \longrightarrow H_2\big(\widetilde{X}, \mathbb{Z}\big) \xrightarrow{\pi_{1*} \oplus \pi_{2*}} \underbrace{H_2\big(B_1, \mathbb{Z}\big) \oplus H_2\big(B_2, \mathbb{Z}\big)}_{= H_2\left(B_1 \times B_2, \mathbb{Z}\right)} \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{3.22}$$

assures us that we can study the curves in $\widetilde{X}$ completely by looking at their image in $B_1 \times B_2$. All we have to do is determine the curves in $B_1 \times B_2$ that lie on the hypersurface $\widetilde{X}$.

Let us introduce the notation

$$C_1 \underline{\times} C_2 \;=\; \big(C_1 \times C_2\big) \cap \widetilde{X} \;\subset\; \widetilde{X} \;\subset\; B_1 \times B_2 \tag{3.23}$$

for two curves $C_1 \subset B_1$ and $C_2 \subset B_2$. For example,

$$\sigma \underline{\times} \theta_{ij} = \{\text{pt.}\} \times \theta_{ij}, \quad \theta_{ij} \underline{\times} \sigma = \theta_{ij} \times \{\text{pt.}\}. \tag{3.24}$$

Also note that, for example, $\sigma \underline{\times} \sigma$ is a section of the Abelian surface fibration $\widetilde{X} \to \mathbb{P}^1$. Using this notation, a basis for $H_2(\widetilde{X}, \mathbb{Z})$ is

$$\begin{aligned} H_2\big(\widetilde{X}, \mathbb{Z}\big) = \operatorname{span}_{\mathbb{Z}} \big\{ & \sigma \underline{\times} f,\ f \underline{\times} \sigma, \\ & \sigma \underline{\times} \theta_{11},\ \sigma \underline{\times} \theta_{21},\ \sigma \underline{\times} \theta_{31},\ \sigma \underline{\times} \theta_{32},\ \sigma \underline{\times} \theta_{41},\ \sigma \underline{\times} \theta_{42}, \\ & \theta_{11} \underline{\times} \sigma,\ \theta_{21} \underline{\times} \sigma,\ \theta_{31} \underline{\times} \sigma,\ \theta_{32} \underline{\times} \sigma,\ \theta_{41} \underline{\times} \sigma,\ \theta_{42} \underline{\times} \sigma, \\ & \sigma \underline{\times} \sigma,\ \mu \underline{\times} \sigma,\ \nu \underline{\times} \sigma,\ \sigma \underline{\times} \mu,\ \sigma \underline{\times} \nu \big\} \simeq \mathbb{Z}^{19}. \end{aligned} \tag{3.25}$$

The group action can now easily be determined from the group action on the base, 3.3, and explicitly written in terms of 19×19 matrices. Again, we will use these matrices computationally in the following, but find it unenlightening to actually write them down here. We denote this representation suggestively as

$$R = H_2\big(\widetilde{X}, \mathbb{Z}\big). \tag{3.26}$$

As we will now show, it is dual to the representation $H_4(\widetilde{X}, \mathbb{Z})$.

3.6 Poincaré duality

We now have defined a priori independent bases on $H_4\big(\widetilde{X},\mathbb{Z}\big)$ and $H_2\big(\widetilde{X},\mathbb{Z}\big)$. But they are related through the intersection pairing

$$H_4\big(\widetilde{X},\mathbb{Z}\big)\times H_2\big(\widetilde{X},\mathbb{Z}\big)\to\mathbb{Z}=H_0\big(\widetilde{X},\mathbb{Z}\big), \tag{3.27}$$

which is one version of Poincaré duality (see A.1). We can explicitly determine the intersection numbers for our two bases in terms of elementary intersection numbers on B_1 and B_2: For any two basis curves $C_1, C_2 \in \{\sigma, f, \theta_{11}, \dots, \theta_{42}, \mu, \nu\}$ and section $s \in \{\sigma, \mu, \nu\}$

$$\big(C_1\underline{\times}\sigma\big)\cdot\big(\pi_1^{-1}C_2\big) = C_1\cdot C_2 = \big(\sigma\underline{\times}C_1\big)\cdot\big(\pi_2^{-1}C_2\big), \tag{3.28}$$

$$\big(\sigma\underline{\times}s\big)\cdot\big(\pi_1^{-1}C_2\big) = s\cdot C_2 = \big(s\underline{\times}\sigma\big)\cdot\big(\pi_2^{-1}C_2\big), \tag{3.29}$$

$$\big(\sigma\underline{\times}C_1\big)\cdot\big(\pi_1^{-1}s\big) = C_1\cdot s = \big(C_1\underline{\times}\sigma\big)\cdot\big(\pi_2^{-1}s\big), \tag{3.30}$$

and 0 in the remaining cases. For example, $(\theta_{11}\underline{\times}\sigma)\cdot(\pi_2^{-1}\theta_{11})=0$.

This makes it easy to write down the explicit 19×19 intersection matrix. One can check that its determinant is 1, as it should be. The inverse matrix is again integral and defines the Poincaré dual of any curve or divisor. In particularly, it follows that R and $R^\vee$, eqs. (3.26) and (3.20), are mutually dual representations, as we already implied by the notation.

3.7 Middle dimension

For completeness, let us also discuss the $G=\mathbb{Z}_3\times\mathbb{Z}_3$-action on the middle dimensional homology group $H_3(\widetilde{X},\mathbb{Z})\simeq\mathbb{Z}^{40}$. By Poincaré duality, this representation must be self-dual. Unfortunately, there seems to be no simple way to write down an integral basis of three-cycles. We did construct a G-CW complex of the 4-skeleton of $\widetilde{X}$, that is, a cell complex on which G acts by permutation of cells. Given this, finding the action on homology boils down to a lengthy linear algebra exercise on the corresponding chain complex. With the help of a computer we found the explicit 40×40 representation matrices for $H3$. As above, we do not write out the explicit matrices but simply define this $\mathbb{Z}_3\times\mathbb{Z}_3$ representation to be

$$H3 = H_3\big(\widetilde{X},\mathbb{Z}\big). \tag{3.31}$$

Note that we will only need information about $H3$ in 5.3, where it could be replaced by some independent toric computation.

4. Properties of the group action

4.1 Describing integer representations

Summarizing the results of 3, the $G=\mathbb{Z}_3\times\mathbb{Z}_3$ group action on the homology and cohomology of $\widetilde{X}$ is

$$H_{6-i}\big(\widetilde{X},\mathbb{Z}\big) = H^i\big(\widetilde{X},\mathbb{Z}\big) = \begin{cases} \mathbb{Z} & i=6 \\ 0 & i=5 \\ R \simeq \mathbb{Z}^{19} & i=4 \\ H3 \simeq \mathbb{Z}^{40} & i=3 \\ R^\vee \simeq \mathbb{Z}^{19} & i=2 \\ 0 & i=1 \\ \mathbb{Z} & i=0, \end{cases} \tag{4.1}$$

where we used Poincaré duality as well, see A.1. Of course, we are really interested in the quotient X and not in the covering space $\widetilde{X}$. However, as we will show is 5, the homology of the quotient X can be calculated from the G-action on the homology of $\widetilde{X}$. More precisely, certain invariants, called *group homology*, of the group action on $H_*(\widetilde{X},\mathbb{Z})$ are the starting point for the Cartan-Leray spectral sequence, which in turn computes $H_*(X,\mathbb{Z})$. Dually, the Leray-Serre spectral sequence computes the cohomology on X from the *group cohomology groups* of the group action on $H^*(\widetilde{X},\mathbb{Z})$. The purpose of this section is to find the group homology groups of the G-representations $H_q(\widetilde{X},\mathbb{Z})$ and group cohomology groups of the G-representations $H^q(\widetilde{X},\mathbb{Z})$. These are denoted by

$$H_p\Big(G, H_q\big(\widetilde{X},\mathbb{Z}\big)\Big), \quad H^p\Big(G, H^q\big(\widetilde{X},\mathbb{Z}\big)\Big). \tag{4.2}$$

An important point is that we are considering representations on integer lattices. Many of the nice features of representation theory on vector spaces no longer hold. In particular, there is no longer any unique decomposition into a sum of irreducible representations. Since the actual integer representations are so complicated, a nice way to classify them is via their group homology and group cohomology. This is entirely analogous to the study of manifolds using their homology and cohomology groups:

Homology and cohomology in topology	Group homology and group cohomology
Manifold X	Group G
Coefficients $C = \mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, twisted coefficients, …	Group representation M
$H_*(X,C)$, $H^*(X,C)$	$H_*(G,M)$, $H^*(G,M)$

An inevitably confusing part of the computation below is that it involves both the "topological homology" and the group homology. Specifically, we need to consider the case where the G-representation is one of the topological homology groups of X. Then, for this representation, we must determine the group homology.

Let us start by defining the group homology and group cohomology. Take any representation M of a finite group G on an integer lattice.[9] In particular, we are interested in the cases where M is either $\mathbb{Z}$ (the trivial representation), R, $R^\vee$, or $H3$. The representation defines a bundle $\widetilde{M}$ of lattices over the classifying space BG through its holonomy around $\pi_1(BG) = G$. The group (co)homology is defined to be the sheaf (co)homology,

$$H_*\big(G, M\big) = H_*\big(BG, \widetilde{M}\big), \quad H^*\big(G, M\big) = H^*\big(BG, \widetilde{M}\big). \tag{4.3}$$

This is a formal, but rather unhelpful definition of group homology and cohomology. However, although defined abstractly via classifying spaces, the actual group homology groups are very computable. All one has to do is compute the cohomology (kernel modulo image) of a certain complex, see [47, 48]. The boundary maps are given explicitly in terms of the G-representation matrices. Computing kernel modulo image then boils down to finding the Smith normal form of the boundary maps, which we calculate using Maple. Basic properties include

- $H^0(G, M) = M^G$, the invariant subspace.
- $H_0(G, M) = M_G$, the coinvariants (See 4.3)
- $H^i(G, M) = 0 = H_i(G, M)$ for $i < 0$.
- $H^i(G, M)$ and $H_i(G, M)$ are *finite* Abelian groups for $i > 0$.

Finally, note that any $\mathbb{Z}_3 \times \mathbb{Z}_3$ representation restricts to a $\mathbb{Z}_3$ representations for each choice of $\mathbb{Z}_3 \subset \mathbb{Z}_3 \times \mathbb{Z}_3$. We are going to need these in the following. Let us write

$$G = \mathbb{Z}_3 \times \mathbb{Z}_3 = G_1 \times G_2 = \{g_1, g_1^2, g_1^3 = 1\} \times \{g_2, g_2^2, g_2^3 = 1\}. \tag{4.4}$$

Of course, there is also a third (diagonal) $\mathbb{Z}_3$ subgroup of $\mathbb{Z}_3 \times \mathbb{Z}_3$, which we denote by $G_{12} = \{1, g_1 g_2, g_1^2 g_2^2\}$. For example, restriction of the $\mathbb{Z}_3 \times \mathbb{Z}_3$-representation R, see eq. (3.26), then defines three $\mathbb{Z}_3$-representations

$$R_1 = R|_{G_1}, \quad R_2 = R|_{G_2}, \quad R_{12} = R|_{G_{12}} \ \in \ \mathbb{Z}_3\text{–Rep} \tag{4.5}$$

corresponding to these three $\mathbb{Z}_3$ subgroups. There are the analogous restrictions of $R^\vee$ and $H3$.

4.2 Invariant cohomology

We start by computing the invariant cohomology of $\widetilde{X}$. This is also the degree zero group cohomology of the topological cohomology of $\widetilde{X}$,

$$H^i\big(\widetilde{X}, \mathbb{Z}\big)^G = H^0\Big(H^i\big(\widetilde{X}, \mathbb{Z}\big)\Big). \tag{4.6}$$

In particular, let us discuss the case $i = 2$. The invariants of a $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ group representation are simple to compute. All one has to do is find the kernel of $\mathrm{id} - g_1$ and

---

[9] M could also have $\mathbb{Z}$-torsion, that is, be of the form $\mathbb{Z}^n \oplus \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_k}$. However the representations we are interested in will be of the form $\mathbb{Z}^n$ only.

id $-g_2$, which is a straightforward linear algebra exercise. For the dP_9 base surfaces, one obtains[10]

$$H^2\big(B_i,\mathbb{Z}\big)^G \simeq H_2\big(B_i,\mathbb{Z}\big)^G = \operatorname{span}\big\{f,\, t\big\} \tag{4.7}$$

where we defined[11]

$$\begin{aligned} t =& \,-3\sigma - 3f + \theta_{11} + \theta_{21} + 2\theta_{31} + 2\theta_{32} + 3\theta_{41} + \theta_{42} + 3\mu + 3\nu \\ =& \,5f + 5\sigma - 2\alpha_1 - \alpha_2 + \alpha_8. \end{aligned} \tag{4.8}$$

On the Calabi-Yau threefold $\widetilde{X}$, the degree-2 invariant cohomology group is then (see [35])

$$H^2\big(\widetilde{X},\mathbb{Z}\big)^G \simeq H_4\big(\widetilde{X},\mathbb{Z}\big)^G = \operatorname{span}\big\{\pi_1^{-1}(f) = \pi_2^{-1}(f),\, \pi_1^{-1}(t),\, \pi_2^{-1}(t)\big\}. \tag{4.9}$$

Let us define the invariant cohomology generators to be[12]

$$\begin{aligned} \phi &= c_1\Big(\mathcal{O}\big(\pi_1^{-1}(f)\big)\Big) = c_1\Big(\mathcal{O}\big(\pi_2^{-1}(f)\big)\Big), \\ \tau_1 &= c_1\Big(\mathcal{O}\big(\pi_1^{-1}(t)\big)\Big), \quad \tau_2 = c_1\Big(\mathcal{O}\big(\pi_2^{-1}(t)\big)\Big) \quad \in H^2\big(\widetilde{X},\mathbb{Z}\big), \end{aligned} \tag{4.10}$$

so that

$$H^2\big(\widetilde{X},\mathbb{Z}\big)^G \simeq H_4\big(\widetilde{X},\mathbb{Z}\big)^G = \operatorname{span}_{\mathbb{Z}}\big\{\phi,\, \tau_1,\, \tau_2\big\}. \tag{4.11}$$

The triple intersection numbers are encoded in the products of ϕ, τ_1, τ_2. One finds that

$$H^{\text{ev}}\big(\widetilde{X},\mathbb{Z}\big)^G = \mathbb{Z}[\tau_1,\tau_2,\phi]\Big/\big\langle \phi^2,\, \tau_1^3,\, \tau_2^3,\, \tau_1\phi = 3\tau_1^2,\, \tau_2\phi = 3\tau_2^2\big\rangle\,. \tag{4.12}$$

Similarly, one can compute the invariant part under the $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ action of all cohomology groups of $\widetilde{X}$. We find that

$$H^0\Big(H^i\big(\widetilde{X},\mathbb{Z}\big)\Big) = H^i\big(\widetilde{X},\mathbb{Z}\big)^G = \begin{cases} \mathbb{Z} & i=6 \\ 0 & i=5 \\ \mathbb{Z}^3 & i=4 \\ \mathbb{Z}^8 & i=3 \\ \mathbb{Z}^3 & i=2 \\ 0 & i=1 \\ \mathbb{Z} & i=0. \end{cases} \tag{4.13}$$

As far as cohomology with real (or complex) coefficients is concerned, the cohomology of the quotient is simply the invariant cohomology on the covering space. That is, for example,

$$H^2\big(\widetilde{X},\mathbb{R}\big)^G = \operatorname{span}_{\mathbb{R}}\big\{\phi,\, \tau_1,\, \tau_2\big\} = \mathbb{R}^3 \quad \Rightarrow \quad H^2\big(X,\mathbb{R}\big) = \mathbb{R}^3, \tag{4.14}$$

and, in particular, $h^{11}(X) = 3$. However, determining the cohomology with integral coefficients on X is far more difficult and will be the subject of 5.

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[10] The middle dimensional homology is self-dual. On B_1, B_2 this is in degree 2. This is why we are not careful in distinguishing the curves on B_i and their Poincaré duals here.

[11] Geometrically, t is the pull-back of the hyperplane divisor via the blow-up map $B_i \to \mathbb{P}^2$.

[12] Again, we explicitly write the identification $H^2 \simeq H_4$ as $c_1\big(\mathcal{O}(-)\big)$. This identification will be implicit in the future.

4.3 Coinvariant homology

The dual notion to invariant cohomology is coinvariant homology, also known as the degree zero group homology group of the homology groups of $\widetilde{X}$,

$$H_i\big(\widetilde{X},\mathbb{Z}\big)_G = H_0\Big(H_i\big(\widetilde{X},\mathbb{Z}\big)\Big). \tag{4.15}$$

Since we are mainly interested in curves, we are going to consider the $i = 2$ case in detail. It turns out that there is a clear reason why the coinvariant curves are of particular interest. To see this, consider the $G = \mathbb{Z}_3 \times \mathbb{Z}_3$-quotient map

$$q : \widetilde{X} \to X. \tag{4.16}$$

This map of manifolds determines the push-forward q_* of homology groups as follows. Pick any 2-cycle $\widetilde{C} \subset \widetilde{X}$, and let us denote its image by $C = q\big(\widetilde{C}\big) \subset X$.

- If $\dim_{\mathbb{R}} C < 2$, then $q_*\big(\widetilde{C}\big) = 0$.
- If $q\big|_{\widetilde{C}} : \widetilde{C} \to C$ is one-to-one, then $q_*\big(\widetilde{C}\big) = C$.
- If $q\big|_{\widetilde{C}} : \widetilde{C} \to C$ is n-to-one, then $q_*\big(\widetilde{C}\big) = nC$.

One tautological property of the push-forward is that

$$q_*\big(\widetilde{C}\big) = q_*\Big(g\big(\widetilde{C}\big)\Big) \quad \forall g \in G,\ \widetilde{C} \in H_2\big(\widetilde{X},\mathbb{Z}\big). \tag{4.17}$$

In other words,

$$q_*\Big(\widetilde{C} - g\big(\widetilde{C}\big)\Big) = 0 \quad \forall g \in G,\ \widetilde{C} \in H_2\big(\widetilde{X},\mathbb{Z}\big). \tag{4.18}$$

Put yet differently, there are obvious relations

$$I = \operatorname{span}_{\mathbb{Z}} \Big\{ \widetilde{C} - g\big(\widetilde{C}\big) \,\Big|\, g \in G,\ \widetilde{C} \in H_2\big(\widetilde{X},\mathbb{Z}\big) \Big\} \subset H_2\big(\widetilde{X},\mathbb{Z}\big) \tag{4.19}$$

that push forward to zero. The quotient by these relations is called the *coinvariant* homology,

$$H_2\big(\widetilde{X},\mathbb{Z}\big)_G = H_2\big(\widetilde{X},\mathbb{Z}\big) \Big/ I. \tag{4.20}$$

The push-forward map obviously factorizes

$$\begin{tikzcd}
H_2\big(\widetilde{X},\mathbb{Z}\big) \arrow[rr, "q_*"] \arrow[dr, "\text{mod } I"'] & & H_2\big(X,\mathbb{Z}\big) \\
& H_2\big(\widetilde{X},\mathbb{Z}\big)_G \arrow[ur, "\hat{q}_*"'] &
\end{tikzcd} \tag{4.21}$$

One nice set of generators for the relations I using the notation of eq. (3.25) is

$$\begin{aligned}
\sigma\underline{\times}\theta_{ij} &= \sigma\underline{\times}\theta_{11} \quad \forall i = 1,2,3,4;\ j = 0,1,2; && (4.22)\\
\theta_{ij}\underline{\times}\sigma &= \theta_{11}\underline{\times}\sigma \quad \forall i = 1,2,3,4;\ j = 0,1,2; && (4.23)\\
\sigma\underline{\times}f &= 3\,\sigma\underline{\times}\theta_{11}, \qquad f\underline{\times}\sigma = 3\,\theta_{11}\underline{\times}\sigma, && (4.24)\\
2\,\sigma\underline{\times}\sigma &= \mu\underline{\times}\sigma + \sigma\underline{\times}\mu, \qquad \sigma\underline{\times}\sigma + \nu\underline{\times}\sigma = 2\,\sigma\underline{\times}\nu, && (4.25)\\
3\big(\sigma\underline{\times}\mu - \sigma\underline{\times}\sigma\big) &= 0, \qquad 3\big(\sigma\underline{\times}\nu - \sigma\underline{\times}\sigma\big) = 0. && (4.26)
\end{aligned}$$

Interestingly, the last two relations can only be obtained with an overall factor of 3, but not without! For example, take

$$\begin{aligned} \widetilde{C}_1 &= 2\,\sigma\underline{\times}\theta_{31} - 2\,\sigma\underline{\times}\theta_{41} + \theta_{21}\underline{\times}\sigma + \theta_{31}\underline{\times}\sigma + 3\,\mu\underline{\times}\sigma - 3\,\nu\underline{\times}\sigma, \\ \widetilde{C}_2 &= 2\,\sigma\underline{\times}\theta_{32} + 2\,\sigma\underline{\times}\theta_{41} - 2\,\theta_{31}\underline{\times}\sigma - \theta_{32}\underline{\times}\sigma - \theta_{41}\underline{\times}\sigma - \theta_{42}\underline{\times}\sigma, \end{aligned} \tag{4.27}$$

then

$$\widetilde{C}_1 - g_1\big(\widetilde{C}_1\big) + \widetilde{C}_2 - g_2\big(\widetilde{C}_2\big) = 3\big(\sigma\underline{\times}\mu - \sigma\underline{\times}\sigma\big). \tag{4.28}$$

We conclude that the coinvariant homology of $\widetilde{X}$ can be written as

$$\begin{aligned} H_2\big(\widetilde{X},\mathbb{Z}\big)_G &= \big(\sigma\underline{\times}\theta_{11}\big)\mathbb{Z} \oplus \big(\theta_{11}\underline{\times}\sigma\big)\mathbb{Z} \oplus \big(\sigma\underline{\times}\sigma\big)\mathbb{Z} \\ &\quad \oplus \big(\sigma\underline{\times}\mu - \sigma\underline{\times}\sigma\big)\mathbb{Z}_3 \oplus \big(\sigma\underline{\times}\nu - \sigma\underline{\times}\sigma\big)\mathbb{Z}_3 \\ &\simeq \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3. \end{aligned} \tag{4.29}$$

Moreover, the push-downs of the generating curves have clear geometric interpretations:

- X is again elliptically fibered over B_1/G_1 and B_2/G_2. The homology class of the fiber is $q_*(\theta_{11}\underline{\times}\sigma)$ and $q_*(\sigma\underline{\times}\theta_{11})$, respectively.
- Due to the two independent elliptic fibrations, X is also fibered by Abelian surfaces $X \to \mathbb{P}^1$. Note that, since the G action on $\widetilde{X}$ is by translation along fibers, it does not act on the base $\mathbb{P}^1$. The zero section is $q_*(\sigma\underline{\times}\sigma)$.
- The torsion curves $q_*(\sigma\underline{\times}\mu - \sigma\underline{\times}\sigma)$ and $q_*(\sigma\underline{\times}\nu - \sigma\underline{\times}\sigma)$ are differences of sections of the Abelian surface fibration.

Similarly to the above, we have computed all of the coinvariant homology groups of $\widetilde{X}$ with respect to $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, and found

$$H_0\Big(H_i\big(\widetilde{X},\mathbb{Z}\big)\Big) = H_i\big(\widetilde{X},\mathbb{Z}\big)_G = \begin{cases} \mathbb{Z} & i=6 \\ 0 & i=5 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_3 & i=4 \\ \mathbb{Z}^8 \oplus \big(\mathbb{Z}_3\big)^4 & i=3 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i=2 \\ 0 & i=1 \\ \mathbb{Z} & i=0. \end{cases} \tag{4.30}$$

Recall that, modulo torsion, the invariant (co)homology of $\widetilde{X}$ is the (co)homology of X. Is the coinvariant homology of $\widetilde{X}$ exactly equal to the homology of the quotient X, including the torsion subgroups? In general, this is not an easy question, and one needs extra generators and extra relations. However, as we will show in 5, in degree 2 the coinvariant homology does capture the whole torsion information, that is

$$\hat{q}_*\Big[\underbrace{H_2\big(\widetilde{X},\mathbb{Z}\big)_{G,\ \text{tors}}}_{=\mathbb{Z}_3\oplus\mathbb{Z}_3}\Big] = H_2\big(X,\mathbb{Z}\big)_{\text{tors}} = \mathbb{Z}_3 \oplus \mathbb{Z}_3. \tag{4.31}$$

On the other hand, the free part $H_2(\widetilde{X},\mathbb{Z})_{G,\ \text{free}} \simeq \mathbb{Z}^3$ does not push down to the whole $H_2(X,\mathbb{Z})$, as we will discuss later in detail.

4.4 Group (co)homology groups

So far, we have only computed the degree 0 group homology and group cohomology groups of the representations R, $R^\vee$, $H3$ in eq. (4.1). However, in order to compute the homology of the quotient X, which will be done in the next section, we also need the higher group homology and group cohomology groups.

Because the case of a cyclic group ($\mathbb{Z}_3$) is simpler, let us first consider the restriction of R, $R^\vee$, $H3$ to different Z_3 subgroups of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$. Since we have the group action given in terms of explicit integer matrices, finding any particular group (co)homology group is just a linear algebra exercise, see 4.1. Combined with the fact that the positive degree cohomology groups of a cyclic group are 2-periodic, this determines all $\mathbb{Z}_3$ group (co)homology groups. We have computed all of these group (co)homology groups, and found that they are

$$\begin{aligned} H^j\big(\mathbb{Z}_3, R_i\big) = H^j\big(\mathbb{Z}_3, R_i^\vee\big) &\simeq \begin{cases} \mathbb{Z}_3 \oplus \mathbb{Z}_3 & j = 2k \\ \mathbb{Z}_3 & j = 2k+1 \\ \mathbb{Z}^7 & j = 0 \end{cases} \\ H_j\big(\mathbb{Z}_3, R_i\big) = H_j\big(\mathbb{Z}_3, R_i^\vee\big) &\simeq \begin{cases} \mathbb{Z}_3 & j = 2k \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 & j = 2k+1 \\ \mathbb{Z}^7 \oplus \mathbb{Z}_3 & j = 0 \end{cases} \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} H^j\big(\mathbb{Z}_3, H3_i\big) = H^j\big(\mathbb{Z}_3, H3_i^\vee\big) &\simeq \begin{cases} \big(\mathbb{Z}_3\big)^6 & j = 2k \\ \big(\mathbb{Z}_3\big)^2 & j = 2k+1 \\ \mathbb{Z}^{16} & j = 0 \end{cases} \\ H_j\big(\mathbb{Z}_3, H3_i\big) = H_j\big(\mathbb{Z}_3, H3_i^\vee\big) &\simeq \begin{cases} \big(\mathbb{Z}_3\big)^2 & j = 2k \\ \big(\mathbb{Z}_3\big)^6 & j = 2k+1 \\ \mathbb{Z}^{16} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & j = 0 \end{cases} \end{aligned} \tag{4.33}$$

independently of whether $i = 1$, 2, or 12.

Finally, we will need the group homology and group cohomology of $\mathbb{Z}_3 \times \mathbb{Z}_3$. We have already determined the degree zero part in Subsections 4.2 and 4.3, but will need some of the higher degrees in the following. They turn out to be

i	0	1	2	3	4	6	$\cdots$
$H_i\big(G, R\big)$	$\mathbb{Z}^3 \oplus (\mathbb{Z}_3)^2$	$(\mathbb{Z}_3)^5$	$(\mathbb{Z}_3)^5$	$(\mathbb{Z}_3)^8$	$(\mathbb{Z}_3)^8$	$(\mathbb{Z}_3)^{11}$	$\cdots$
$H_i\big(G, R^\vee\big)$	$\mathbb{Z}^3 \oplus \mathbb{Z}_3$	$(\mathbb{Z}_3)^4$	$(\mathbb{Z}_3)^4$	$(\mathbb{Z}_3)^7$	$(\mathbb{Z}_3)^7$	$(\mathbb{Z}_3)^{10}$	$\cdots$
$H^i\big(G, R\big)$	$\mathbb{Z}^3$	$\mathbb{Z}_3$	$(\mathbb{Z}_3)^4$	$(\mathbb{Z}_3)^4$	$(\mathbb{Z}_3)^7$	$(\mathbb{Z}_3)^7$	$\cdots$
$H^i\big(G, R^\vee\big)$	$\mathbb{Z}^3$	$(\mathbb{Z}_3)^2$	$(\mathbb{Z}_3)^5$	$(\mathbb{Z}_3)^5$	$(\mathbb{Z}_3)^8$	$(\mathbb{Z}_3)^8$	$\cdots$
$H_i\big(G, H3\big)$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_3)^4$	$(\mathbb{Z}_3)^{12}$	$(\mathbb{Z}_3)^9$	$(\mathbb{Z}_3)^{17}$	$(\mathbb{Z}_3)^{14}$	$(\mathbb{Z}_3)^{22}$	$\cdots$
$H^i\big(G, H3\big)$	$\mathbb{Z}^8$	$(\mathbb{Z}_3)^4$	$(\mathbb{Z}_3)^{12}$	$(\mathbb{Z}_3)^9$	$(\mathbb{Z}_3)^{17}$	$(\mathbb{Z}_3)^{14}$	$\cdots$

(4.34)

Interestingly, this proves that the representation R is not isomorphic to its dual.

5. Homology and cohomology

5.1 General form

We now have all the information necessary to compute the homology and cohomology groups with integer coefficients on $\widetilde{X}/\mathbb{Z}_3$ and $\widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. However, since this involves many mathematical details, we first preview the results. The non-mathematically oriented reader is advised to peruse this subsection only, skipping the remainder of 5.

We begin by considering the integral homology groups. As we have already mentioned, the rank of the integral homology of the quotient is determined by the rank of the coinvariant homology of $\widetilde{X}$. For $X/\mathbb{Z}_3$, this can be read off from the degree-0 group homology groups ($j = 0$) in eqs. (4.32) and (4.33). Similarly, the $i = 0$ column in eq. (4.34) provides this information for $X = \widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. However, this only determines the free part of the homology of X and gives us no information on the torsion part, which must be computed in another way. Note that, although there are in principle seven non-vanishing homology groups on a 6-dimensional manifold, only four of them can contain a torsion subgroup. Moreover, using Poincaré duality and the Universal Coefficient Theorem, there are only two distinct torsion subgroups, each occurring twice in the homology of the 6-dimensional manifold [49]. In our case, one of the torsion subgroups is simply determined from the group action and the ensuing fundamental groups $\pi_1(\widetilde{X}/\mathbb{Z}_3) = \mathbb{Z}_3$ and $\pi_1(X) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$. We denote the remaining unknown finite subgroup by T_3 and T_{33}, respectively. Putting all of this information together, the integral homology of the quotients must be of the form

$$H_i\big(\widetilde{X}/\mathbb{Z}_3, \mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z} \\ 0 \\ \mathbb{Z}^7 \oplus \mathbb{Z}_3 \\ \mathbb{Z}^{16} \oplus T_3 \\ \mathbb{Z}^7 \oplus T_3 \\ \mathbb{Z}_3 \\ \mathbb{Z}, \end{cases} \qquad H_i\big(\widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3), \mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z} & i = 6 \\ 0 & i = 5 \\ \mathbb{Z}^3 \oplus \big(\mathbb{Z}_3\big)^2 & i = 4 \\ \mathbb{Z}^8 \oplus T_{33} & i = 3 \\ \mathbb{Z}^3 \oplus T_{33} & i = 2 \\ \big(\mathbb{Z}_3\big)^2 & i = 1 \\ \mathbb{Z} & i = 0. \end{cases} \tag{5.1}$$

In the remainder of this section, we are going to compute T_3 and T_{33}. The result will be that

$$T_3 \simeq \mathbb{Z}_3, \quad T_{33} \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3. \tag{5.2}$$

In fact, we can be more precise and identify the geometry of the torsion curves. We will see that the torsion curves are images of curves on the covering space $\widetilde{X}$, something that is not automatic. Explicitly, the push-forward by the quotient maps $\hat{q} : \widetilde{X} \to \widetilde{X}/\mathbb{Z}_3$ and $q : \widetilde{X} \to X$ is an isomorphism

$$\begin{aligned} \hat{q}_* : H_2\big(\widetilde{X}, \mathbb{Z}\big)_{\mathbb{Z}_3,\text{tors}} &\xrightarrow{\sim} H_2\big(\widetilde{X}/\mathbb{Z}_3, \mathbb{Z}\big)_{\text{tors}}, \\ q_* : H_2\big(\widetilde{X}, \mathbb{Z}\big)_{G,\text{tors}} &\xrightarrow{\sim} H_2\big(X, \mathbb{Z}\big)_{\text{tors}} \end{aligned} \tag{5.3}$$

between the torsion parts of coinvariant homology on $\widetilde{X}$ and the homology on the quotient. Note that the free parts of the respective homology groups are equal as well, raising the obvious question whether the push-forward is an isomorphism for the whole integral homology. For the intermediate quotient, $\widetilde{X}/\mathbb{Z}_3$, this is indeed so and

$$\hat{q}_*:\ H_2\big(\widetilde{X},\mathbb{Z}\big)_{\mathbb{Z}_3} \xrightarrow{\sim} H_2\big(\widetilde{X}/\mathbb{Z}_3,\mathbb{Z}\big). \tag{5.4}$$

However, on X there is the following subtlety. The degree-2 homology classes on any simply connected manifold, for example $\widetilde{X}$, can always be represented by spheres and, therefore, the image of q_* is a linear combination of spheres. But on X not every degree-2 homology class can be represented by spheres. To make this more precise, we denote the spherical homology classes by $\Sigma_2(X,\mathbb{Z})$. A convenient definition is to start with $\pi_2(X)$, the second homotopy group of X, and look at its image in homology, that is,

$$\Sigma_2(X,\mathbb{Z}) = \operatorname{img}\big[\pi_2(X)\big] \subset H_2(X,\mathbb{Z}). \tag{5.5}$$

In our case, it turns out that

$$\begin{aligned} \Sigma_2\big(\widetilde{X}/\mathbb{Z}_3,\mathbb{Z}\big) &= H_2\big(\widetilde{X}/\mathbb{Z}_3,\mathbb{Z}\big), \\ \Sigma_2\big(X,\mathbb{Z}\big)_{\text{tors}} &= H_2\big(X,\mathbb{Z}\big)_{\text{tors}}, \end{aligned} \tag{5.6}$$

while

$$\Sigma_2\big(X,\mathbb{Z}\big)_{\text{free}} \subsetneq H_2\big(X,\mathbb{Z}\big)_{\text{free}} \tag{5.7}$$

is a sublattice of index 3. To summarize, the push-forward by the quotient maps actually is an isomorphism

$$\hat{q}_*:\ H_2\big(\widetilde{X},\mathbb{Z}\big)_{\mathbb{Z}_3} \xrightarrow{\sim} \Sigma_2\big(\widetilde{X}/\mathbb{Z}_3,\mathbb{Z}\big), \qquad q_*:\ H_2\big(\widetilde{X},\mathbb{Z}\big)_G \xrightarrow{\sim} \Sigma_2\big(X,\mathbb{Z}\big), \tag{5.8}$$

between the coinvariant homology and the homology classes that are representable by linear combinations of spheres. Since we are only interested in the genus 0 worldsheet instantons for the purposes of this paper, we actually only need Σ_2 and not H_2.

As a final remark, note that X is a non-toric example where the mirror symmetry conjecture of [3] holds: Let Y and Y^* be a pair of mirror Calabi-Yau threefolds. Then it is conjectured[13] that

$$H_1\big(Y,\mathbb{Z}\big)_{\text{tors}} = H_2\big(Y^*,\mathbb{Z}\big)_{\text{tors}}. \tag{5.9}$$

Previously [3], this has been checked for the 16 toric hypersurfaces with non-trivial fundamental group. In those 16 cases $H_1\big(Y,\mathbb{Z}\big)_{\text{tors}} = \pi_1(Y)$ is non-trivial while $H_2\big(Y,\mathbb{Z}\big)_{\text{tors}} = 0$, and their mirror manifolds satisfy the above relation. In our case, X is ,presumably, self-mirror and, in contrast to the toric hypersurface case, its mirror is again a free quotient. The homology of X again satisfies the above mirror relation $H_1\big(X,\mathbb{Z}\big)_{\text{tors}} = T_{33} = H_2\big(X,\mathbb{Z}\big)_{\text{tors}}$.

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[13] This mirror conjecture can be written in terms of integral cohomology as well. The equivalent statement then is $H^2(Y,\mathbb{Z})_{\text{tors}} = H^3(Y^*,\mathbb{Z})_{\text{tors}}$.

5.2 Spectral sequences

We are now going to compute the remaining unknown torsion subgroups T_3, T_{33} in eq. (5.1). To do so, we will rely on two spectral sequences which we will review below. Applying one of these spectral sequences in 5.3, we will compute the integral cohomology of $\widetilde{X}/\mathbb{Z}_3$. Using the other spectral sequence, we will then attempt to compute $H_2(X,\mathbb{Z})$ in 5.4 and find that there are two possible answers. Finally, in 5.5, we resolve this ambiguity and determine the integral homology and cohomology of X.

The cohomology version of the aforementioned spectral sequences is [50, 51]

Theorem 1 (Leray-Serre spectral sequence). *For any manifold Y with free*[14] *G action, there is a cohomology spectral sequence*

$$E_2^{p,q} = H^p\Big(G, H^q(Y,\mathbb{Z})\Big) \quad \Longrightarrow \quad H^{p+q}\Big(Y/G,\mathbb{Z}\Big). \tag{5.10}$$

In particular, $E_2^{0,q} = H^q(Y,\mathbb{Z})^G$ is the invariant cohomology.

The analogous sequence for homology groups is [52]

Theorem 2 (Cartan-Leray spectral sequence). *For any manifold Y with free G action, there is a homology spectral sequence*

$$E^2_{p,q} = H_p\Big(G, H_q(Y,\mathbb{Z})\Big) \quad \Longrightarrow \quad H_{p+q}\Big(Y/G,\mathbb{Z}\Big). \tag{5.11}$$

In particular, $E^2_{0,q} = H_q(Y,\mathbb{Z})_G$ is the coinvariant homology.

Hence, the Cartan-Leray spectral sequence describes the precise relationship between coinvariant homology and the homology of the quotient. Dually, the Leray-Serre spectral sequence describes the precise relationship between invariant cohomology and the cohomology of the quotient.

5.3 The partial quotient

As a warm-up exercise, and since we are going to need some of these results in the following, we begin with the computation of the cohomology of the partial quotient $\widetilde{X}/G_i$, where $G_i \simeq \mathbb{Z}_3$ (see 4.1). It turns out that nothing depends on whether we consider G_1, G_2, or G_{12}, so we need not make any distinction between them in this subsection. Note that, while the $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action is not toric, any single $\mathbb{Z}_3$ subgroup can be chosen to act only by phase multiplications. For example, in the coordinates used in eqs. (2.2a) and (2.2b), the g_1 action, eq. (2.3a), is toric. Hence, the partial quotient can also be treated using toric methods, see section 4 in Part B [30]. In particular, its integral homology groups could be computed as in [3].

We use the Leray-Serre spectral sequence to compute the cohomology of X/G_i starting from the G_1 group action on the cohomology of $\widetilde{X}$. The E_2 tableau consists of the group

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[14] More generally, this spectral sequence computes the G-equivariant cohomology. For free group actions, this is the same as the cohomology of the quotient.

cohomology groups computed in eqs. (4.32) and (4.33),

$$E_2^{p,q}\big(\widetilde{X}/G_i\big) = \begin{tikzcd}
q{=}6 & \mathbb{Z} & 0 & \mathbb{Z}_3 & 0 & \mathbb{Z}_3 & 0 & \mathbb{Z}_3 & 0 & \cdots \\
q{=}5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}4 & \mathbb{Z}^7 & \mathbb{Z}_3 & \mathbb{Z}_3^2 \arrow[drr] & \mathbb{Z}_3 \arrow[drr] & \mathbb{Z}_3^2 & \mathbb{Z}_3 & \mathbb{Z}_3^2 & \mathbb{Z}_3 & \cdots \\
q{=}3 & \mathbb{Z}^{16} & \mathbb{Z}_3^2 & \mathbb{Z}_3^6 & \mathbb{Z}_3^2 & \mathbb{Z}_3^6 \arrow[drr] & \mathbb{Z}_3^2 \arrow[drr] & \mathbb{Z}_3^6 & \mathbb{Z}_3^2 & \cdots \\
q{=}2 & \mathbb{Z}^7 & \mathbb{Z}_3 & \mathbb{Z}_3^2 & \mathbb{Z}_3 & \mathbb{Z}_3^2 & \mathbb{Z}_3 & \mathbb{Z}_3^2 & \mathbb{Z}_3 & \cdots \\
q{=}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}0 & \mathbb{Z} & 0 & \mathbb{Z}_3 & 0 & \mathbb{Z}_3 & 0 & \mathbb{Z}_3 & 0 & \cdots \\
 & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 & p{=}6 & p{=}7 &
\end{tikzcd} \; . \tag{5.12}$$

The E_2 tableau is obviously not bounded to the right. However, in the E_∞ tableau all entries with $p+q > 6$ have to vanish since $H^{p+q}\big(\widetilde{X}/\mathbb{Z}_3, \mathbb{Z}\big) = 0$ if $p+q > 6$. Hence, the superfluous entries must be removed by higher differentials. Since the E_2 tableau is 2-periodic for sufficiently large p, we first consider the case where every differential starts or ends in the periodic range. Counting the ranks of possible differentials, the entries can only be completely removed if every non-zero differential either starts or ends in the $q = 3$ row. And, moreover, each such differential starting or ending at $q = 3$ must have maximal rank.

This argument determines all differentials for sufficiently large p, but we also need the differentials for small p. Note that the cohomology Leray-Serre spectral sequence is actually a spectral sequence of $H^*(\mathbb{Z}_3, \mathbb{Z})$-algebras. Therefore, the differentials

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} \tag{5.13}$$

for $p \gg 0$ are all induced from $d_r^{0,q}$, $d_r^{1,q}$, and multiplication with the generator in $E_r^{2,0}$. Hence we know all d_2 differentials, not only the ones with $p \gg 0$. Therefore, we determine the next tableau to be

$$E_3^{p,q} = \begin{tikzcd}
q{=}6 & \mathbb{Z} & 0 & \mathbb{Z}_3 & 0 & \mathbb{Z}_3 & 0 & \mathbb{Z}_3 & 0 & \cdots \\
q{=}5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}4 & \mathbb{Z}^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}3 & \mathbb{Z}^{16} & \mathbb{Z}_3 & \mathbb{Z}_3^2 & 0 & \mathbb{Z}_3^2 & 0 & \mathbb{Z}_3^2 & 0 & \cdots \\
q{=}2 & \mathbb{Z}^7 & \mathbb{Z}_3 \arrow[ddrrr, "d_3"] & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}0 & \mathbb{Z} & 0 & \mathbb{Z}_3 & 0 & \mathbb{Z}_3 & 0 & \mathbb{Z}_3 & 0 & \cdots \\
 & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 & p{=}6 & p{=}7 &
\end{tikzcd} \; . \tag{5.14}$$

The d_3 drawn above must vanish, since the range has to survive until $d_4^{0,3} : \mathbb{Z}^{16} \to \mathbb{Z}_3$.

Hence, $E_3^{p,q} = E_4^{p,q}$ and the d_4-cohomology is

$$E_5^{p,q} = E_\infty^{p,q} = \begin{array}{c|ccccccccc}
q{=}6 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}4 & \mathbb{Z}^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}3 & \mathbb{Z}^{16} & \mathbb{Z}_3 & \mathbb{Z}_3 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}2 & \mathbb{Z}^7 & \mathbb{Z}_3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}0 & \mathbb{Z} & 0 & \mathbb{Z}_3 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
 & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 & p{=}6 & p{=}7 &
\end{array} . \tag{5.15}$$

Looking at the diagonals, there are no extension ambiguities and we can read off the cohomology. The Universal Coefficient Theorem then fixes the homology. The result is

$$H^i\big(\widetilde{X}/\mathbb{Z}_3, \mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z} \\ \mathbb{Z}_3 \\ \mathbb{Z}^7 \oplus \mathbb{Z}_3 \\ \mathbb{Z}^{16} \oplus \mathbb{Z}_3 \\ \mathbb{Z}^7 \oplus \mathbb{Z}_3 \\ 0 \\ \mathbb{Z} \end{cases} \quad \Rightarrow \quad H_i\big(\widetilde{X}/\mathbb{Z}_3, \mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z} & i=6 \\ 0 & i=5 \\ \mathbb{Z}^7 \oplus \mathbb{Z}_3 & i=4 \\ \mathbb{Z}^{16} \oplus \mathbb{Z}_3 & i=3 \\ \mathbb{Z}^7 \oplus \mathbb{Z}_3 & i=2 \\ \mathbb{Z}_3 & i=1 \\ \mathbb{Z} & i=0. \end{cases} \tag{5.16}$$

Hence, we have determined T_3 in eq. (5.1) to be

$$T_3 \simeq \mathbb{Z}_3. \tag{5.17}$$

Now that we know the result, let us return to the corresponding Cartan-Leray spectral sequence. The bottom part of the E^3 tableau is

$$E^3_{p,q}\big(\widetilde{X}/G_i\big) = \begin{tikzcd}
q{=}2 & \mathbb{Z}^7 \oplus \mathbb{Z}_3 & \vdots & \vdots & \vdots & \vdots & \iddots \\
q{=}1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}0 & \mathbb{Z} & \mathbb{Z}_3 & 0 & \mathbb{Z}_3 \arrow[llluu, "d^3_{(i)}"'] & 0 & \cdots \\
 & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & \cdots
\end{tikzcd} . \tag{5.18}$$

From the cohomology computation, we know that the torsion curve $\mathbb{Z}_3$ has to survive[15] to

$$H_2\big(\widetilde{X}/G_i, \mathbb{Z}\big) = H_2\big(\widetilde{X}, \mathbb{Z}\big)_{G_i} \simeq \mathbb{Z}^7 \oplus \mathbb{Z}_3. \tag{5.19}$$

Hence, the above differential

$$d^3_{(i)} : \; E^3_{3,0}\big(\widetilde{X}/G_i\big) \xrightarrow{0} E^3_{0,2}\big(\widetilde{X}/G_i\big) \tag{5.20}$$

must vanish. We will need this result in the following.

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[15] That is, must not be removed by differentials or extensions.

5.4 The full quotient

We now compute the degree-2 homology groups of $X = \widetilde{X}/G$ with $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ using the Cartan-Leray spectral sequence. The bottom part, which does not depend on d_2, is

$$\begin{array}{r} E^2_{p,q}\big(\widetilde{X}/G\big) = \\ E^3_{p,q}\big(\widetilde{X}/G\big) = \end{array} \begin{tikzcd} q{=}2 & \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \vdots & \vdots & \vdots & \vdots & \iddots \\ q{=}1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ q{=}0 & \mathbb{Z} & \big(\mathbb{Z}_3\big)^2 & \mathbb{Z}_3 & \big(\mathbb{Z}_3\big)^3 \arrow[uulll, bend right, "d^3"'] & \big(\mathbb{Z}_3\big)^2 & \cdots \\ & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & \cdots \end{tikzcd} . \tag{5.21}$$

Knowing the differential in the $\widetilde{X}/G_i$ spectral sequence above, we can determine the differential d^3 in the $\widetilde{X}/G$ spectral sequence as follows. The quotient map

$$q_i : \widetilde{X}/G_i \longrightarrow \widetilde{X}/G \tag{5.22}$$

induces a morphism of spectral sequences

$$q_{i*} : \Big\{ E^r_{\bullet,\bullet}\big(\widetilde{X}/G_i\big), d^r_{(i)} \Big\} \longrightarrow \Big\{ E^r_{\bullet,\bullet}\big(\widetilde{X}/G\big), d^r \Big\} . \tag{5.23}$$

In particular, for $r = 3$ there is a commutative diagram

$$\begin{tikzcd} \mathbb{Z}_3 \simeq E^3_{3,0}\big(\widetilde{X}/G_i\big) \arrow[r, "d^3_{(i)}=0"] \arrow[d, "G_i \subset G"] & E^3_{0,2}\big(\widetilde{X}/G_i\big) \simeq \mathbb{Z}_3 \oplus \mathbb{Z}^7 \arrow[d, "q_{i*}"] \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \simeq E^3_{3,0}\big(\widetilde{X}/G\big) \arrow[r, "d^3"] & E^3_{0,2}\big(\widetilde{X}/G\big) \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}^3 . \end{tikzcd} \tag{5.24}$$

The $E^3_{p,0}$ terms are just group homology, and only depend on the group. It is fairly clear that the inclusion $G_1 \subset G$ and $G_2 \subset G$ map onto two of the three $\mathbb{Z}_3$ summands in $H_3(G;\mathbb{Z})$. A bit of homological algebra, see C, shows that the inclusion of the diagonal $G_{12} \subset G$ then maps onto the third summand. So we can find 3 generators of $E^3_{3,0}\big(\widetilde{X}/G\big) = H_3(G,\mathbb{Z})$ which are induced from some $E^3_{3,0}\big(\widetilde{X}/G_i\big)$. Moreover,

$$q_{i*} : \underbrace{H_2\big(\widetilde{X},\mathbb{Z}\big)_{G_i}}_{=E^3_{0,2}\left(\widetilde{X}/G_i\right)} \longrightarrow \underbrace{H_2\big(\widetilde{X},\mathbb{Z}\big)_{G}}_{=E^3_{0,2}\left(\widetilde{X}/G\right)} \tag{5.25}$$

is surjective, since enlarging the group only adds more relations to the coinvariant homology. Therefore, commutativity forces

$$d^3 = 0 . \tag{5.26}$$

To summarize, we found that the following entries in the tableau eq. (5.21) survive to $r = \infty$,

$$E^\infty_{p,q}\big(\widetilde{X}/G\big) = \begin{array}{r|llll} q{=}2 & \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \vdots & \vdots & \iddots \\ q{=}1 & 0 & 0 & 0 & \cdots \\ q{=}0 & \mathbb{Z} & \big(\mathbb{Z}_3\big)^2 & \mathbb{Z}_3 & \cdots \\ \hline & p{=}0 & p{=}1 & p{=}2 & \cdots \end{array} . \tag{5.27}$$

Having determined the endpoint of the Cartan-Leray spectral sequence for $\widetilde{X}/G$, we still do not quite know its homology. We have to solve one extension ambiguity, which takes the form of the short exact sequence

$$0 \longrightarrow \underbrace{H_2\big(\widetilde{X}, \mathbb{Z}\big)_G}_{\simeq \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3} \xrightarrow{q_*} H_2\big(\widetilde{X}/G, \mathbb{Z}\big) \longrightarrow \underbrace{H_2\big(G, \mathbb{Z}\big)}_{\simeq \mathbb{Z}_3} \longrightarrow 0\,, \tag{5.28}$$

where the first map q_* is just the pushforward by the $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient map

$$q : \widetilde{X} \longrightarrow \widetilde{X}/G = X. \tag{5.29}$$

Depending on which extension is realized, the homology group could either be

$$H_2\big(X, \mathbb{Z}\big) \simeq \mathbb{Z}^3 \oplus \big(\mathbb{Z}_3\big)^2 \quad \text{or} \quad \mathbb{Z}^3 \oplus \big(\mathbb{Z}_3\big)^3. \tag{5.30}$$

This leaves two possibilities, either $T_{33} = (\mathbb{Z}_3)^2$ or $T_{33} = (\mathbb{Z}_3)^3$, for the torsion group in eq. (5.1). In the next subsection, we will fix this ambiguity.

5.5 A higher differential and final result

Recall that there is also a Leray-Serre spectral sequence for the cohomology of the quotient $X = \widetilde{X}/G$. Its E_2 tableau reads

$$E_2^{p,q}\big(\widetilde{X}/G\big) = \begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
q{=}3 & \mathbb{Z}^8 & \mathbb{Z}_3^4 & \mathbb{Z}_3^{12} & \mathbb{Z}_3^9 & \mathbb{Z}_3^{17} & \mathbb{Z}_3^{14} & \cdots \\
q{=}2 & \mathbb{Z}^3 \arrow[ddrrr, "d_3"] & \mathbb{Z}_3^2 & \mathbb{Z}_3^5 & \mathbb{Z}_3^5 & \mathbb{Z}_3^8 & \mathbb{Z}_3^8 & \cdots \\
q{=}1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q{=}0 & \mathbb{Z} & 0 & \mathbb{Z}_3^2 & \mathbb{Z}_3 & \mathbb{Z}_3^3 & \mathbb{Z}_3^2 & \cdots \\
 & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 & \cdots
\end{tikzcd} . \tag{5.31}$$

With this in mind, there are two dual ways of fixing the ambiguity encountered in the previous subsection:

1. Identify the short exact sequence eq. (5.28) with the sequence [53, 54]

   $$0 \longrightarrow \Sigma_2\big(\widetilde{X}/G, \mathbb{Z}\big) \hookrightarrow H_2\big(\widetilde{X}/G, \mathbb{Z}\big) \longrightarrow H_2\big(G, \mathbb{Z}\big) \longrightarrow 0, \tag{5.32}$$

   where Σ_2 are the homology classes of degree 2 which are representable by spheres, see eq. (5.5). If one can find a higher genus holomorphic curve in $\widetilde{X}/G$ whose homology class is not representable by spheres, then the short exact sequence does not split. This way to fix the ambiguity was used in [54] for a certain quotient of the quintic.

2. If the differential $d_3 : E_3^{0,2} \to E_3^{3,0}$ in eq. (5.31) is non-trivial, then $E_\infty^{3,0} = 0$ and the torsion part $H^3\big(\widetilde{X}/G, \mathbb{Z}\big)_{\text{tors}}$ is at most $E_2^{1,2} = (\mathbb{Z}_3)^2$. Hence the second possibility in eq. (5.30) would be ruled out, fixing the ambiguity.

We will follow the latter route and compute

$$d_3 : \ \underbrace{H^2\big(\widetilde{X},\mathbb{Z}\big)^G}_{\simeq\mathbb{Z}^3} \longrightarrow \underbrace{H^3\big(G,\mathbb{Z}\big)}_{\simeq\mathbb{Z}_3} . \tag{5.33}$$

Note that we can identify two key objects with certain line bundles on $\widetilde{X}$. Recall the correspondence between $H^2\big(\widetilde{X},\mathbb{Z}\big)$ and line bundles via the first Chern class, 3.4:

- $H^2\big(\widetilde{X},\mathbb{Z}\big)^G$ are the G-invariant line bundles.
- Evaluating the Leray-Serre spectral sequence, eq. (5.31), yields

  $$\ker\big(d_3\big) \oplus \big(\mathbb{Z}_3\big)^2 = \left[\bigoplus_{p+q=2} E_\infty^{p,q}\right] = H^2\big(X,\mathbb{Z}\big). \tag{5.34}$$

  Pulling back to $\widetilde{X}$ via the quotient map kills the torsion part $(\mathbb{Z}_3)^2$, and we obtain

  $$q^*\Big[H^2(X,\mathbb{Z})\Big] = \ker\big(d_3\big) \subset H^2\big(\widetilde{X},\mathbb{Z}\big)^G \subset H^2\big(\widetilde{X},\mathbb{Z}\big) \tag{5.35}$$

  But the pull-backs of line bundles on the quotient $X = \widetilde{X}/G$ are precisely the G-equivariant line bundles on $\widetilde{X}$. Hence, $\ker\big(d_3\big)$ are the G-equivariant line bundles.

The differential d_3 is either zero or surjective. Therefore, $\ker(d_3)$ is either all of $H^2(\widetilde{X},\mathbb{Z})^G$ or an index-3 sublattice, respectively. In fact, the latter is true:

Example 1. *Consider the line bundle*

$$\mathcal{O}_{\widetilde{X}}\big(\tau_i\big) = \mathcal{O}_{\widetilde{X}}\Big(\pi_i^{-1}(t)\Big) = \pi_i^*\Big(\mathcal{O}_{B_i}(t)\Big) \tag{5.36}$$

on $\widetilde{X}$, which is pulled back from one of the base dP_9 surfaces B_i. This line bundle is G-invariant but not G-equivariant.

Proof. The line bundle is invariant because $\pi_i^{-1}(t)$ is an invariant divisor class, see eq. (4.9). It remains to show that the line bundle is not equivariant. Assume, on the contrary, that $\pi_i^*\big(\mathcal{O}_{B_i}(t)\big)$ were equivariant. Then

$$\pi_{i*}\Big[\pi_i^*\Big(\mathcal{O}_{B_i}(t)\Big)\Big] = \mathcal{O}_{B_i}(t) \tag{5.37}$$

would be equivariant, and hence $\mathcal{O}_{B_i}(t)|_f = \mathcal{O}_f(t\cdot f) = \mathcal{O}_f\big(3\{\text{pt.}\}\big)$ would be G-equivariant. But $G \simeq \mathbb{Z}_3\times\mathbb{Z}_3$ acts on $f \simeq T^2$ by two independent order-3 translations, so any equivariant bundle must have degree divisible by 9. Hence the degree 3 line bundle $\mathcal{O}_f(t\cdot f)$ cannot be equivariant, contradicting our assumption. ☐

To summarize, the differential d_3 had to remove the invariant-but-not-equivariant line bundles when descending to X and ,hence, had to be nontrivial. Therefore, the torsion part $H^3\big(\widetilde{X},\mathbb{Z}\big)_{\text{tors}} \simeq H_2\big(\widetilde{X},\mathbb{Z}\big)_{\text{tors}}$ in eq. (5.30) can be at most $(\mathbb{Z}_3)^2$ and, therefore,

$$H_2\big(X,\mathbb{Z}\big) \simeq \mathbb{Z}^3 \oplus \big(\mathbb{Z}_3\big)^2, \quad H^3\big(X,\mathbb{Z}\big) \simeq \mathbb{Z}^8 \oplus \big(\mathbb{Z}_3\big)^2 . \tag{5.38}$$

It follows that we have determined T_{33} in eq. (5.1) to be

$$T_{33} \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3. \tag{5.39}$$

This fixes the last ambiguity in the integral homology and cohomology of X. The final result is

$$H^i\big(X,\mathbb{Z}\big) = H_{6-i}\big(X,\mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z} & i=6 \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i=5 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i=4 \\ \mathbb{Z}^8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i=3 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 & i=2 \\ 0 & i=1 \\ \mathbb{Z} & i=0. \end{cases} \tag{5.40}$$

Part II

Instantons

6. Quotients of the quintic

6.1 Curves and Kähler classes

Having found the complete integral homology and cohomology groups including torsion, we turn to the second topic of this paper, that is, computing the Gromov-Witten invariants, or instanton numbers, on $X = \widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. We begin by reviewing the simpler and well-studied case of the quintic Calabi-Yau threefold and its $\mathbb{Z}_5$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$ quotients. Although the quintic and its quotients do not have torsion curves, we will encounter some subtleties associated with the group quotients that are also relevant to our case.

In particular, consider the one-parameter family

$$Q = \left\{ z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \psi^5 z_0 z_1 z_2 z_3 z_4 = 0 \right\} \subset \mathbb{P}^4 \tag{6.1}$$

of quintic threefolds. The defining equation is invariant under the $\mathbb{Z}_5 \times \mathbb{Z}_5 \subset PGL(5, \mathbb{C})$ group action

$$\begin{aligned} [z_0 : z_1 : z_2 : z_3 : z_4] &\mapsto [z_1 : z_2 : z_3 : z_4 : z_0] \\ [z_0 : z_1 : z_2 : z_3 : z_4] &\mapsto [z_0 : e^{\frac{2\pi i}{5}} z_1 : e^{\frac{4\pi i}{5}} z_2 : e^{\frac{6\pi i}{5}} z_3 : e^{\frac{8\pi i}{5}} z_4]. \end{aligned} \tag{6.2}$$

The group action has fixed points in $\mathbb{P}^4$, but they do not lie on the hypersurface Q. Hence, the quotients[16] $Q/\mathbb{Z}_5$ and $Q/\big(\mathbb{Z}_5 \times \mathbb{Z}_5\big)$ are smooth Calabi-Yau threefolds. Let us put a bar over quantities on the $\mathbb{Z}_5$ quotient and use a double bar for the $\mathbb{Z}_5 \times \mathbb{Z}_5$ quotient,

$$\bar{Q} = Q/\mathbb{Z}_5, \qquad \bar{\bar{Q}} = Q \Big/ \big(\mathbb{Z}_5 \times \mathbb{Z}_5\big) = \bar{Q}/\mathbb{Z}_5. \tag{6.3}$$

The rational cohomology is always one-dimensional in each even degree, generated by the hyperplane class of the ambient $\mathbb{P}^4$. However, if one keeps track of the proper normalization, things are slightly more complicated. Moreover, there are torsion 1-cycles corresponding to the discrete Wilson lines on the quotients.

Recall that $h^{11}(Q) = 1$ and $h^{21}(Q) = 101$. Note specifically that there is only a single Kähler modulus. Thus, while the odd degree cohomology groups are fairly large, the even degree cohomology, that is $H^{\text{ev}} = H^0 \oplus H^2 \oplus H^4 \oplus H^6$, is very manageable. For the quintic and its quotients they are

$$H^{\text{ev}}\big(Q, \mathbb{Z}\big) = \mathbb{Z}[\xi_2, \xi_4] \Big/ \big\langle \xi_2^2 = 5\xi_4,\ (\dim > 6)\ \big\rangle \tag{6.4a}$$

$$H^{\text{ev}}\big(\bar{Q}, \mathbb{Z}\big) = \mathbb{Z}\left[\bar{\xi}_2, \bar{\tau}_2\right] \Big/ \big\langle 5\bar{\tau}_2,\ \bar{\tau}_2^2,\ \bar{\tau}_2\bar{\xi}_2,\ (\dim > 6)\ \big\rangle \tag{6.4b}$$

$$\begin{aligned} H^{\text{ev}}\left(\bar{\bar{Q}}, \mathbb{Z}\right) = \mathbb{Z}\left[\bar{\bar{\xi}}_2, \bar{\bar{\tau}}_2, \bar{\bar{\rho}}_2, \bar{\bar{\xi}}_4, \bar{\bar{\xi}}_6\right] \Big/ \big\langle & 5\bar{\bar{\tau}}_2,\ 5\bar{\bar{\rho}}_2,\ \bar{\bar{\tau}}_2^2,\ \bar{\bar{\tau}}_2\bar{\bar{\rho}}_2,\ \bar{\bar{\rho}}_2^2, \\ & \bar{\bar{\tau}}_2\bar{\bar{\xi}}_2,\ \bar{\bar{\tau}}_2\bar{\bar{\xi}}_4,\ \bar{\bar{\rho}}_2\bar{\bar{\xi}}_2,\ \bar{\bar{\rho}}_2\bar{\bar{\xi}}_4, \\ & \bar{\bar{\xi}}_2^2 = 5\bar{\bar{\xi}}_4,\ \bar{\bar{\xi}}_2\bar{\bar{\xi}}_4 = 5\bar{\bar{\xi}}_6,\ (\dim > 6)\ \big\rangle, \end{aligned} \tag{6.4c}$$

---

[16] Of course, there are 6 different $\mathbb{Z}_5$ subgroups in $\mathbb{Z}_5 \times \mathbb{Z}_5$. However, that distinction will not be relevant in the following.

where the subscripts on the generators are their dimension and we do not explicitly write the relations imposed by dimension > 6 terms. Note the appearance of torsion classes $\bar{\tau}_2$, $\bar{\bar{\tau}}_2$, and $\bar{\bar{\rho}}_2$. These are the first Chern classes of flat line bundles (the Wilson lines).

The pull backs under the successive quotients can be determined by computing the higher differentials in the Leray-Serre spectral sequence. This is tedious but straightforward, and we will not present the details. One finds that

$$\begin{tikzcd}
 & & \overbrace{\qquad}^{H^0} & & & & \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{H^2} & & & & \overbrace{\qquad\quad}^{H^4} & & \overbrace{\qquad\quad}^{H^6} \\
H^{\mathrm{ev}}(Q,\mathbb{Z}) & = & \mathbb{Z} & \oplus & \xi_2\mathbb{Z} & \oplus & 0 & & \oplus & & \xi_4\mathbb{Z} & \oplus & \xi_2\xi_4\mathbb{Z} \\
H^{\mathrm{ev}}\left(\bar{Q},\mathbb{Z}\right) & = & \mathbb{Z} \arrow[u, "\times 1"'] & \oplus & \bar{\xi}_2\mathbb{Z} \arrow[u, "\times 1"'] & \oplus & \bar{\tau}_2\mathbb{Z}_5 \arrow[u] & \oplus & 0 & \oplus & \bar{\xi}_2^2\mathbb{Z} \arrow[u, "\times 5"'] & \oplus & \bar{\xi}_2^3\mathbb{Z} \arrow[u, "\times 5"'] \\
H^{\mathrm{ev}}\left(\bar{\bar{Q}},\mathbb{Z}\right) & = & \mathbb{Z} \arrow[u, "\times 1"'] & \oplus & \bar{\bar{\xi}}_2\mathbb{Z} \arrow[u, "\times 5"'] & \oplus & \bar{\bar{\tau}}_2\mathbb{Z}_5 \arrow[u, "\times 1"'] & \oplus & \bar{\bar{\rho}}_2\mathbb{Z}_5 \arrow[u] & \oplus & \bar{\bar{\xi}}_4\mathbb{Z} \arrow[u, "\times 5"'] & \oplus & \bar{\bar{\xi}}_6\mathbb{Z},
\end{tikzcd} \tag{6.5}$$

where we picked integral generators in each even cohomology group. By separating the different degrees, one can easily read off any even cohomology group. For example, $H^2(\bar{Q},\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_5$ and it is generated by $\bar{\xi}_2$ and $\bar{\tau}_2$. We observe that there is only a single Kähler modulus on Q, $\bar{Q}$, and $\bar{\bar{Q}}$. However, when comparing them there is a subtlety involving the correct integral normalization. The integral generator $\bar{\xi}_2$ pulls back to the integral generator ξ_2, while the integral generator $\bar{\bar{\xi}}_2$ pulls back to five times the integral generator $\bar{\xi}_2$.

The corresponding Poincaré dual push downs in homology are

$$\begin{tikzcd}
 & & \overbrace{\qquad\qquad}^{H_0} & & \overbrace{\qquad\quad}^{H_2} & & & & \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{H_4} & & & & \overbrace{\qquad\quad}^{H_6} \\
H_{\mathrm{ev}}(Q,\mathbb{Z}) & = & \{\mathrm{pt.}\}\mathbb{Z} \arrow[d, "\times 1"'] & \oplus & C\mathbb{Z} \arrow[d, "\times 1"'] & \oplus & 0 \arrow[d] & & \oplus & & D\mathbb{Z} \arrow[d, "\times 5"'] & \oplus & QZ \arrow[d, "\times 5"'] \\
H_{\mathrm{ev}}\left(\bar{Q},\mathbb{Z}\right) & = & \{\mathrm{pt.}\}\mathbb{Z} \arrow[d, "\times 1"'] & \oplus & \bar{C}\mathbb{Z} \arrow[d, "\times 5"'] & \oplus & \bar{\tau}_4\mathbb{Z}_5 \arrow[d, "\times 1"'] & \oplus & 0 \arrow[d] & \oplus & \bar{D}\mathbb{Z} \arrow[d, "\times 5"'] & \oplus & \bar{Q}\mathbb{Z} \arrow[d, "\times 5"'] \\
H_{\mathrm{ev}}\left(\bar{\bar{Q}},\mathbb{Z}\right) & = & \{\mathrm{pt.}\}\mathbb{Z} & \oplus & \bar{\bar{C}}\mathbb{Z} & \oplus & \bar{\bar{\tau}}_4\mathbb{Z}_5 & \oplus & \bar{\bar{\rho}}_4\mathbb{Z}_5 & \oplus & \bar{\bar{D}}\mathbb{Z} & \oplus & \bar{\bar{Q}}\mathbb{Z},
\end{tikzcd} \tag{6.6}$$

where C, $\bar{C}$, $\bar{\bar{C}}$ and D, $\bar{D}$, $\bar{\bar{D}}$ are generating curves[17] and divisors, respectively. Furthermore, we denote by $\bar{\bar{\tau}}_4$, $\bar{\bar{\rho}}_4$ and $\bar{\tau}_4$ the torsion generators in $H_4(\bar{Q},\mathbb{Z})$ and $H_4(\bar{\bar{Q}},\mathbb{Z})$. We observe again that, while the curve classes are abstractly the same 1-dimensional lattice

$$H_2\big(Q,\mathbb{Z}\big) \simeq H_2\big(\bar{Q},\mathbb{Z}\big) \simeq H_2\big(\bar{\bar{Q}},\mathbb{Z}\big) \simeq \mathbb{Z}, \tag{6.7}$$

---

[17] C and $\bar{C}$ can be taken to be rational curves, whereas the homology class of $\bar{\bar{C}}$ can *not* be represented by a rational curve [54]. $\bar{\bar{C}}$ can be represented by a genus 1 curve.

the normalization of the curves is subtle. The $\mathbb{Z}_5$-quotient of the generator C is again a generator, but the $\mathbb{Z}_5$-quotient of the generator $\bar{C}$ is five times a generator in $H_2(\bar{\bar{Q}}, \mathbb{Z})$.

6.2 Instantons on the quintic

We now turn to the worldsheet instanton corrections to certain Yukawa couplings. To be more precise, we consider the $E_8 \times E_8$ heterotic string on the quintic Q (and, similarly, $\bar{Q}$, $\bar{\bar{Q}}$) with the standard embedding. This choice of gauge bundle breaks $E_8 \to E_6$. Recall that the massless E_6 matter fields correspond to the bundle-valued cohomology groups

$$H^1\big(Q, TQ\big), \qquad\qquad H^1\big(Q, TQ^\vee\big) = H^1\big(Q, \Omega_Q\big) = H^{1,1}(Q) = \xi_2\mathbb{C} \tag{6.8}$$

for the $\underline{\mathbf{27}}$ and $\overline{\underline{\mathbf{27}}}$ representations, respectively. Conveniently, there is a single $\overline{\underline{\mathbf{27}}}$ matter field corresponding to $H^1(Q, TQ^\vee)$ and we will only consider its Yukawa couplings. These can be computed by calculating a three-point function in the A-model[18] topological string. More precisely, the harmonic form associated with the generator $\xi_2 \in H^{1,1}(Q)$ corresponds to a chiral operator $\mathcal{O}_{\xi_2}$ in the conformal field theory. Classically, the Yukawa coupling is just the triple overlap integral of ξ_2, or, equivalently, the triple intersection number of the Poincaré dual divisor. The result is that

$$\big\langle \mathcal{O}^3_{\xi_2} \big\rangle_{\text{classical}} = \int_Q \xi_2 \wedge \xi_2 \wedge \xi_2 = \int_Q 5\xi_2 \wedge \xi_4 = 5, \tag{6.9}$$

where we used the relation eq. (6.4a) and the fact that $\xi_2 \wedge \xi_4$ is the properly normalized volume form. Due to a non-renormalization theorem, there are no perturbative corrections. However, genus 0 worldsheet instantons can and do contribute. The triumph of mirror symmetry was that this duality allows one to actually calculate the instanton effects. For example, the correctly normalized three-point function for the quintic turns out to be [1]

$$\big\langle \mathcal{O}^3_{\xi_2} \big\rangle = 5 + 2875q + 4876875q^2 + \cdots\,, \tag{6.10a}$$

where $q = e^{2\pi i t}$ is the minimal instanton action. Similarly, the three-point function for the $\mathbb{Z}_5$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$ quotient are given by [54]

$$\big\langle \bar{\mathcal{O}}^3_{\bar{\xi}_2} \big\rangle = 1 + 575q + 975375q^2 + \cdots\,, \tag{6.10b}$$

$$\big\langle \bar{\bar{\mathcal{O}}}^3_{\bar{\bar{\xi}}_2} \big\rangle = 25 + 14375q^5 + 24384375q^{10} + \cdots\,. \tag{6.10c}$$

To count the number of instantons n_d of volume d, one has to compare these results with the formal q-series for the instanton-corrected Yukawa coupling. This has the general form [1]

$$\big\langle \mathcal{O}^3 \big\rangle = \kappa_{111} + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d}, \tag{6.11}$$

---

[18] Conversely, the Yukawa couplings of the fields coming from $H^1(Q, TQ)$ are a three-point function in the B-model.

where κ_{111} is the triple intersection number. Note that each minimal curve can be wrapped multiply times, contributing at different volumes. In the instanton expansion above, this is already taken into account by the factor

$$\frac{q^d}{1-q^d} = q^d + q^{2d} + q^{3d} + \cdots = \sum_{i=1}^{\infty} q^{id}. \tag{6.12}$$

Comparing the instanton-corrected three-point functions in eqs. (6.10a), (6.10b), and (6.10c) to the general form of the instanton series eq. (6.11), we can read of the non-vanishing instanton numbers

Q	$\bar{Q}$	$\bar{\bar{Q}}$
$\kappa_{111} = 5$	$\bar{\kappa}_{111} = 1$	$\bar{\bar{\kappa}}_{111} = 25$
$n_1 = 2875$	$\bar{n}_1 = 575 = \frac{n_1}{5}$	$\bar{\bar{n}}_5 = 115 = \frac{n_1}{25}$
$n_2 = 609250$	$\bar{n}_2 = 121850 = \frac{n_2}{5}$	$\bar{\bar{n}}_{10} = 24370 = \frac{n_2}{25}$
$n_3 = 317206375$	$\bar{n}_3 = 63441275 = \frac{n_3}{5}$	$\bar{\bar{n}}_{15} = 12688255 = \frac{n_3}{25}$
$\vdots$	$\vdots$	$\ddots$

(6.13)

We make two important observations, both of which apply to $X = \widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ as well:

- The number of rational curves on the quotient of some freely acting group G is $\frac{1}{|G|}$ times the number of corresponding rational curves on the covering space.
- Even if a curve class is primitive (not a multiple of another curve) on the covering space, its image on the quotient can still be non-primitive.

To summarize, we first computed the relations between the degree-2 homology and cohomology in the quintic Q and its quotients $\bar{Q}$, $\bar{\bar{Q}}$. This allows one to compute the classical $\underline{\overline{\mathbf{27}}}^3$ Yukawa couplings. The classical result on the quintic can be extended to the complete worldsheet instanton corrected three-point functions using mirror symmetry. By comparing the resulting instanton expansion with the formal q-series of the Yukawa couplings, one can read off the instanton numbers on the covering space Q. The corresponding instanton numbers on $\bar{Q}$, $\bar{\bar{Q}}$ are $\frac{1}{5}$ and $\frac{1}{25}$, respectively, of the instanton numbers on Q. This last result is true for all free quotients, and will be used in the following.

Having established these results, we now warn the reader that we will *not* continue to work with the Yukawa couplings. Rather, we will calculate the genus 0 prepotential instead. For the quintic, this amounts to the triple integral over the Kähler modulus t,

$$\mathcal{F}_{Q,0}(q) = \iiint \left\langle \mathcal{O}^3_{\xi_2} \right\rangle \mathrm{d}t^3 = \frac{1}{3!}\kappa_{111} t^3 + p_2(t) + \underbrace{\frac{1}{(2\pi \mathrm{i})^3} \sum_{d=1}^{\infty} n_d \,\mathrm{Li}_3(q^d)}_{=\mathcal{F}^{\mathrm{np}}_{Q,0}(q)}, \tag{6.14}$$

where $p_2(t)$ is a quadratic polynomial and $\mathrm{Li}_3(q) = \sum_{n=1}^{\infty} \frac{q^n}{n^3}$ takes care of multi-covers of the same curve. Clearly, the non-perturbative part $\mathcal{F}^{\mathrm{np}}_{Q,0}(q)$ of the prepotential contains the same information about the instanton numbers as the three-point functions. The real advantage of this formulation is that there is always only one prepotential, whereas, for example on the 19-parameter Calabi-Yau $\widetilde{X}$, there would be $\binom{19+3-1}{3} = 1330$ three-point functions. On a general Calabi-Yau threefold, Y, with $r = h^{11}(Y)$ Kähler moduli $t^1, \dots, t^r$, the prepotential is of the form

$$\begin{aligned} \mathcal{F}_{Y,0}(q_1, \dots, q_r) = \frac{1}{3!} \sum_{1 \leq a \leq b \leq c \leq r} \kappa_{abc} t^a t^b t^c + p_2(t^1, \dots, t^r) \\ + \underbrace{\frac{1}{(2\pi \mathrm{i})^3} \sum_{d_1, \dots, d_r} n_{(d_1, \dots, d_r)} \, \mathrm{Li}_3 \left(\prod_{i=1}^{r} q_i^{d_i} \right)}_{= \mathcal{F}^{\mathrm{np}}_{Y,0}(q_1, \dots, q_r)}, \end{aligned} \tag{6.15}$$

where $q_i = e^{2\pi \mathrm{i} t^i}$. The three-point functions can be recovered as

$$\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_\ell \rangle = \partial_{t^i} \partial_{t^j} \partial_{t^\ell} \mathcal{F}_{Y,0}(q_1, \dots, q_r). \tag{6.16}$$

7. A-model on the covering space $\widetilde{\mathbf{X}}$

7.1 Curves

We now return to the main objective of this paper, which is to compute the instanton numbers (Gromov-Witten invariants) for the Calabi-Yau threefold X defined in 2. However, before graduating to the non-simply connected X, we first have to understand the universal cover $\widetilde{X}$. Fortunately, a generic Schoen Calabi-Yau threefold, that is, the fiber product of two generic dP_9 surfaces, was studied in [29]. Using the E_8 Mordell-Weil group of a generic dP_9, they expressed the prepotential in terms of E_8 theta functions, see also [55]. Our covering space $\widetilde{X}$ is such a Schoen Calabi-Yau threefold, although one with a special $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry. In our case, the Mordell-Weil groups are just $MW(B_i) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$. However, although the actual curves change[19] as we move to a $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetric point in the complex structure moduli space, the instanton numbers do not jump. So we might just as well use the instanton numbers computed for generic complex structure moduli.

In the remainder of this subsection, we will review the above A-model computation. Let $\hat{B}_1$, $\hat{B}_2$ be two generic dP_9 surfaces ($12 I_0$ Kodaira fibers), and define the fiber product

$$\hat{X} = \hat{B}_1 \times_{\mathbb{P}^1} \hat{B}_2. \tag{7.1}$$

The surfaces $\hat{B}_i$ now have infinitely many sections forming the E_8 root lattice

$$MW\big(\hat{B}_i\big) \simeq \Lambda_{E_8} = \left(\left\{ \mathop{\boxplus}_{i=1}^{8} \big(\boxplus_{n_i} \alpha_i \big) \,\middle|\, n_i \in \mathbb{Z} \right\}, \langle -, - \rangle \right), \tag{7.2}$$

---

[19] This phenomenon is already familiar from the quintic, for which there are 375 isolated curves and 50 one-parameter families at the Fermat point, while generically all $2875 = 5 \cdot 375 + 20 \cdot 50$ are isolated.

where we will use the notation of 3.2 for a choice of simple roots. The Calabi-Yau threefold $\hat{X} \to \mathbb{P}^1$ is fibered by Abelian surfaces, so we again have a group law on the sections. This defines the group

$$MW\big(\hat{X}\big) = \Big\{ s_1 \underline{\times} s_2 \;\Big|\; s_1 \in MW(\hat{B}_1), s_2 \in MW(\hat{B}_2) \Big\} = MW\big(\hat{B}_1\big) \oplus MW\big(\hat{B}_2\big). \qquad (7.3)$$

Now we can describe part of the rational curves in $\hat{X}$:

- Vertical curves[20] are precisely the components of singular fibers. The Abelian surface fibration $\hat{X} \to \mathbb{P}^1$ has 12 singular fibers of type $I_0 \times T^2$ and 12 singular fibers of type $T^2 \times I_0$, so there are 24 families. The moduli space $\mathcal{M}_{\text{Vert}}$ of each family is a T^2, so $\chi\big(\mathcal{M}_{\text{Vert}}\big) = 0$ and they do not contribute to the instanton numbers.

- The sections in $MW(\hat{X})$ are the only *smooth* rational curves s with $s \cdot \phi = 1$.

- Each (smooth) section s passes through the singular fibers of $\hat{X} \to \mathbb{P}^1$. Pick, for example, one such $I_0 \times T^2$. Amongst the one-parameter family of I_0, there is precisely one I_0^s which intersects s. Therefore, $s \cup I_0^s$ is an isolated (reducible) rational curve. Those curves are called *pseudo-sections* in [29], and all curves C with $C \cdot \phi = 1$ are either sections or of this form.

- Multi-sections, that is, curves C with $C \cdot \phi \geq 2$, are not yet understood.

These curves contribute to the instanton numbers with some (integral) multiplicity. Roughly, the multiplicity is the Euler characteristic of the moduli space of the curve (this needs to be refined if the moduli space is singular). Hence,

- The moduli space $\mathcal{M}_{\text{Vert}}$ of each vertical curve is a T^2, so $\chi\big(\mathcal{M}_{\text{Vert}}\big) = 0$ and they do not contribute to the instanton numbers.

- Sections do not have infinitesimal deformations, $N_{s|\hat{X}} = \mathcal{O}_s(-1) \oplus \mathcal{O}_s(-1)$. Hence, they contribute to the instanton numbers with multiplicity 1. The volume of such a section is

  $$V_s = \int_s J = s \cdot J, \qquad (7.4)$$

  where $J \in H^2\big(\hat{X}, \mathbb{R}\big)$ is the Kähler form.

- Consider a pseudo-section P consisting of a section s and covering the i-th Kodaira fiber m_i times. Then it contributes to the instanton numbers with a pre-factor (see [29, 56])

  $$n(P) = \prod_{i=1}^{24} p\big(m_i\big), \qquad (7.5)$$

---

[20] In other words, curves that project to a point in the base $\mathbb{P}^1$. Put differently, curves C such that $C \cdot \phi = 0$, where ϕ is the T^4 fiber, see eq. (4.10).

where $p(k)$ is the number of partitions of $k \in \mathbb{Z}_{\geq}$. By definition, the homology class of a pseudo-section is

$$P = s + \sum_{i=1}^{12} m_i \big(f \underline{\times} \sigma\big) + \sum_{i=13}^{24} m_i \big(\sigma \underline{\times} f\big), \tag{7.6}$$

where we labeled the Kodaira fibers such that the first 12 are in the first fiber direction and the remaining 12 are in the other fiber direction. Hence, the volume of a general pseudo-section is

$$V_P = \int_P J = \int_{P_s} J \; + \; \sum_{i=1}^{12} m_i \int_{f \underline{\times} \sigma} J \; + \; \sum_{i=13}^{24} m_i \int_{\sigma \underline{\times} f} J. \tag{7.7}$$

7.2 Prepotential

Using the above knowledge about the curves, one can directly write down their non-perturbative contribution to the prepotential [29]. One obtains

$$\begin{split} \mathcal{F}^{\mathrm{np}}_{\widetilde{X},0} = \sum_{\substack{s_1 \underline{\times} s_2 \\ \in MW(\hat{X})}} e^{2\pi \mathrm{i} \int_{s_1 \underline{\times} s_2} \omega} \left(\sum_{m=0}^{\infty} p(m) e^{2\pi \mathrm{i} m \int_{f \underline{\times} \sigma} \omega} \right)^{12} \left(\sum_{n=0}^{\infty} p(n) e^{2\pi \mathrm{i} n \int_{\sigma \underline{\times} f} \omega} \right)^{12} \\ + \left(\text{contribution of curves with } C \cdot \phi \geq 2\right) \end{split} \tag{7.8}$$

for the genus zero contribution to the prepotential on $\widetilde{X}$, where $\omega = B + \mathrm{i}J$ is the complexified Kähler form. Note that multi-covers of a pseudo-section contribute at the same order as multi-sections, which is why we did not need to include the Li_3 accounting for multi-covers at order p.

Let us define coordinates t^a on the 19-dimensional Kähler moduli space as

$$\omega = t^1 \phi + t^2 \left(\pi_1^{-1} \sigma\right) + \sum_{i=1}^{8} t^{i+2} \big(\pi_1^{-1} \alpha_i\big) + t^{11} \left(\pi_2^{-1} \sigma\right) + \sum_{i=1}^{8} t^{i+11} \big(\pi_2^{-1} \alpha_i\big), \tag{7.9}$$

where we used the basis for the cohomology adapted to the E_8 lattice given in eq. (3.10). In addition, define the Fourier-transformed coordinates

$$\begin{aligned} p_0 &= e^{2\pi \mathrm{i} t^1} = e^{2\pi \mathrm{i} \int_{PD(\phi)} \omega}, \\ q_0 &= e^{2\pi \mathrm{i} t^2}, \quad q_1 = e^{2\pi \mathrm{i} t^3}, \ldots, \; q_8 = e^{2\pi \mathrm{i} t^{10}}, \\ r_0 &= e^{2\pi \mathrm{i} t^{11}}, \quad r_1 = e^{2\pi \mathrm{i} t^{12}}, \ldots, \; r_8 = e^{2\pi \mathrm{i} t^{19}}. \end{aligned} \tag{7.10}$$

It follows that

$$\begin{aligned} e^{2\pi \mathrm{i} \int_{f \underline{\times} \sigma} \omega} &= \prod_{i=0}^{8} q_i, \qquad e^{2\pi \mathrm{i} \int_{\sigma \underline{\times} f} \omega} = \prod_{i=0}^{8} r_i, \\ e^{2\pi \mathrm{i} \int_{s_1 \underline{\times} s_2} \omega} &= p_0 q_0^{s_1 \cdot \sigma} \prod_{i=1}^{8} q_i^{s_1 \cdot \alpha_i} r_0^{s_1 \cdot \sigma} \prod_{i=1}^{8} r_i^{s_2 \cdot \alpha_i}, \end{aligned} \tag{7.11}$$

and, hence,

$$\mathcal{F}^{\mathrm{np}}_{\widetilde{X},0} = p_0 \left(\sum_{\substack{s_1 \in \\ MW(\hat{B}_1)}} q_0^{s_1\cdot\sigma} \prod_{i=0}^{8} q_i^{s_1\cdot\alpha_i} \right) \left(\sum_{\substack{s_2 \in \\ MW(\hat{B}_2)}} r_0^{s_2\cdot\sigma} \prod_{i=0}^{8} r_i^{s_2\cdot\alpha_i} \right) \times$$
$$\times \left(\sum_{m=0}^{\infty} p(m) \prod_{i=0}^{8} q_i^{m} \right)^{12} \left(\sum_{n=0}^{\infty} p(n) \prod_{i=0}^{8} r_i^{n} \right)^{12} + O(p_0^2). \qquad (7.12)$$

Finally, we note the appearance of the generating function for partitions,

$$P(q) = \sum_{i=0}^{\infty} p(i) q^i = \frac{q^{\frac{1}{24}}}{\eta\left(\frac{1}{2\pi\mathrm{i}} \ln q\right)}, \qquad (7.13)$$

and the E_8 theta function[21] (using eq. (3.9))

$$\Theta_{E_8}(q_0; q_1, \ldots, q_8) = \sum_{\gamma\in\Lambda_{E_8}} q_0^{\frac{1}{2}\langle\gamma,\gamma\rangle} \prod_{i=1}^{8} q_i^{\langle\gamma,\alpha_i\rangle} = \sum_{s\in MW(\hat{B})} q_0^{\sigma\cdot s+1} \prod_{i=1}^{8} q_i^{1+s\cdot\sigma-s\cdot\alpha_i}. \qquad (7.14)$$

Therefore,

$$\mathcal{F}^{\mathrm{np}}_{\widetilde{X},0}(p_0, q_0, \ldots, q_8, r_0, \ldots, r_8) = \frac{p_0}{q_0 r_0} \, \widetilde{A}(q_0, \ldots, q_8) \widetilde{A}(r_0, \ldots, r_8) + O(p_0^2), \qquad (7.15)$$

where we defined the auxiliary function

$$\widetilde{A}(q_0, \ldots, q_8) = \Theta_{E_8}\left(\prod_{i=0}^{8} q_i; q_1^{-1}, \ldots, q_8^{-1} \right) P\left(\prod_{i=0}^{8} q_i \right)^{12} \qquad (7.16)$$

and the analogous expression for $\widetilde{A}(r_0, \ldots, r_8)$. Note the occurrence of negative powers of $q_0, \ldots, q_8, r_0, \ldots, r_8$. This is simply an artifact of working in a basis that is adapted to the E_8 lattice structure. In a basis adapted to the Mori cone and the Kähler cone, only positive powers will appear. Nevertheless, by expanding the expression for the prepotential as a series in the 19 variables p_0, $q_0, \ldots, q_8$, $r_0, \ldots, r_8$ and comparing this with the general form eq. (6.15), one can read of the instanton numbers on $\widetilde{X}$. Clearly, the instanton numbers will be indexed by 19 different degrees, making this expansion very cumbersome. Hence, we will refrain from presenting them explicitly.

8. A-model for quotients

8.1 Instantons and the path integral

Before delving into the actual computation of the prepotential and instanton numbers on the quotients of $\widetilde{X}$, we need to understand the effect of torsion homology classes on the

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[21] Usually, the theta function is written as $\Theta_{E_8}(\tau_0; \tau_1, \ldots, \tau_8)$ with $q_i = e^{2\pi\mathrm{i}\tau_i}$. However, we will use our notation since we are going to work with the Fourier-transformed variables everywhere.

instanton sum. The worldsheet instantons in question for an arbitrary Calabi-Yau threefold Y are holomorphic maps $\gamma : \Sigma \to Y$ from the string worldsheet Σ to the target space Y. The path integral sums over all such curves. If we ignore torsion in the homology for a moment, then the effect of an instanton is to add a factor

$$e^{\mathrm{i}S}\big[\gamma : \Sigma \to Y\big] = e^{2\pi \mathrm{i} \int_\Sigma \gamma^* \omega} \tag{8.1}$$

to the path integral, where S is the instanton action and

$$\omega = B + \mathrm{i}J = \sum_a \big(B + \mathrm{i}J\big)^a \, e_a \quad \in H^2\big(Y, \mathbb{C}\big) \tag{8.2}$$

is the complexified Kähler class[22] expanded in some suitable basis $\{e_a\}$ of harmonic forms. Changing variables to

$$q_a = e^{2\pi \mathrm{i}(B+\mathrm{i}J)^a}, \tag{8.3}$$

the instanton factor can be written as

$$e^{\mathrm{i}S}[\gamma] = \prod_a q_a^{d_a} \tag{8.4}$$

with exponents

$$d_a = \int_\Sigma e_a \quad \in \mathbb{Z}_{\geq}. \tag{8.5}$$

Here and everywhere else we assume that the chosen basis $\{e_a\}$ is suitably normalized and, therefore, the exponents d_a are integers.

Now, let us assume that $H_2(Y, \mathbb{Z})$ contains some non-zero torsion part. Since everything said so far only depends only on the integral $\int_\Sigma$, one might at first think that the torsion part of the homology class $\Sigma \in H_2(Y, \mathbb{Z})$ does not enter the path integral at all. However, there is one fallacy in the above reasoning, namely, that the B-field need not be globally defined. So, strictly speaking, the integral $\int_\Sigma B$ is not defined. The correct way is to think about the instanton factor for a flat B-field, $\mathrm{d}B = 0$, as a map assigning to each worldsheet a non-zero complex number[23]

$$e^{\mathrm{i}S} : \; H_2\big(Y, \mathbb{Z}\big) \to \mathbb{C}^\times, \tag{8.6}$$

which can only be written in terms of an integral if one is willing to ignore a subtlety. This subtlety [2] is that the homology classes can have torsion, that is,

$$H_2\big(Y, \mathbb{Z}\big) = H_2\big(Y, \mathbb{Z}\big)_{\text{free}} \oplus H_2\big(Y, \mathbb{Z}\big)_{\text{tors}} = \mathbb{Z}^r \oplus \Big(\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}\Big), \tag{8.7}$$

where r is the rank and the m_i, $i = 1, \ldots, k$ are the torsion coefficients. If there is no torsion, that is, $k = 0$, then the above description is perfectly valid. However, in general one needs in addition to the free generators

$$q_a \;\in \mathrm{Hom}\Big[H_2\big(Y, \mathbb{Z}\big)_{\text{free}}, \mathbb{C}^\times\Big], \qquad a = 1, \ldots, r \tag{8.8}$$

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[22] Since we are really using topological strings on a Calabi-Yau threefold, there cannot be any flux. That is, we require that $\mathrm{d}B = 0$ for the purposes of this paper.

[23] By definition, $\mathbb{C}^\times = \mathbb{C} - \{0\}$ as a multiplicative group.

JHEP10(2007)022

the torsion generators

$$b_i \ \in \mathrm{Hom}\Big[H_2\big(Y,\mathbb{Z}\big)_{\mathrm{tors}},\mathbb{C}^\times\Big], \qquad i=1,\dots,k, \tag{8.9}$$

where

$$b_i^{m_i} = 1. \tag{8.10}$$

In terms of this basis, the instanton factor must be expanded to

$$e^{\mathrm{i}S}[\gamma] = \prod_{a=1}^{r} q_a^{d_a} \ \prod_{i=1}^{k} b_i^{\delta_i} \tag{8.11}$$

with integral exponents

$$d_a \in \big\{0,1,2,\dots\big\}, \quad \delta_i \in \big\{0,\dots,m_i-1\big\}, \tag{8.12}$$

provided that the basis q_a, b_i is correctly normalized. This describes the contribution of any given instanton to the path integral. The non-perturbative correction to the prepotential, see eq. (6.15), generalizes in the obvious way to

$$\begin{aligned}\mathcal{F}_{Y,0}(q_1,\dots,q_r,\,b_1,\dots,b_k) = \frac{1}{3!}\sum_{1\leq a\leq b\leq c\leq r} &\kappa_{abc}t^a t^b t^c + p_2(t^1,\dots,t^r) \\ &+ \underbrace{\frac{1}{(2\pi \mathrm{i})^3}\sum_{\substack{d_1,\dots,d_r\\ \delta_1,\dots,\delta_k}} n_{(d_1,\dots,d_r,\,\delta_1,\dots,\delta_k)}\,\mathrm{Li}_3\left(\prod_{a=1}^{r} q_a^{d_a}\prod_{i=1}^{k} b_i^{\delta_i}\right)}_{=\mathcal{F}^{\mathrm{np}}_{Y,0}(q_1,\dots,q_r,\,b_1,\dots,b_k)},\end{aligned} \tag{8.13}$$

Finally, let us remark on the proper normalization. In principle, the normalization of the q_a, b_i has to be such that they form an integral basis for Hom $\big[H_2(Y,\mathbb{Z}),\mathbb{C}^\times\big]$. However, since we are only considering the genus 0 instantons in the following, one need only consider curve classes that are representable by spheres. Therefore, we will use generators

$$\begin{aligned} q_a \ &\in \mathrm{Hom}\Big[\Sigma_2\big(Y,\mathbb{Z}\big)_{\mathrm{free}},\mathbb{C}^\times\Big], \quad a=1,\dots,r, \\ b_i \ &\in \mathrm{Hom}\Big[\Sigma_2\big(Y,\mathbb{Z}\big)_{\mathrm{tors}},\mathbb{C}^\times\Big], \quad i=1,\dots,k, \end{aligned} \tag{8.14}$$

see eq. (5.5). These are more practical for our purposes, but keep in mind that they might have to be subdivided to write the higher genus prepotential, as we saw in 6. Since we will be interested in the prepotential for $\widetilde{X}$ and two of its quotients, we list the names for the generators eq. (8.14) in 1. We refer the reader to the respective sections for detailed definitions.

8.2 Quotienting the A-model on $\widetilde{\mathbf{X}}$

We finally have everything in place to compute the prepotential on the quotient $X=\widetilde{X}/G$. On general grounds, the $G=\mathbb{Z}_3\times\mathbb{Z}_3$-orbits of a $\mathbb{P}^1\subset\widetilde{X}$ must be $|G|$=9 distinct rational curves since there is no fixed-point free holomorphic map $\mathbb{P}^1\to\mathbb{P}^1$. Hence, there is a one-to-one correspondence between one rational curve on X and a set of $|G|$ rational curves on $\widetilde{X}$, permuted by G.

Therefore, to compute the genus 0 prepotential on the quotient X, we should

Calabi-Yau threefold	r	Free generators	$\{m_1,\dots,m_k\}$	Torsion generators
$\widetilde{X}$	19	$\{p_0,q_0,\dots,q_8,r_0,\dots,r_8\}$	$\varnothing$	$\varnothing$
$\overline{X}=\widetilde{X}/G_1$	7	$\{P,Q_1,Q_2,Q_3,R_1,R_2,R_3\}$	$\{3\}$	$\{b_1\}$
$X=\widetilde{X}/G$	3	$\{p,q,r\}$	$\{3,3\}$	$\{b_1,b_2\}$

Table 1: Variables used in this paper to expand the prepotential for different Calabi-Yau threefolds.

1. Start with the prepotential on $\widetilde{X}$. For the purposes of this subsection, we consider only the terms linear in p_0. This part of the prepotential was computed in eq. (7.8).

2. Impose the relations
$$e^{2\pi \mathrm{i}\int_{\widetilde{C}}\omega}=e^{2\pi \mathrm{i}\int_{g(\widetilde{C})}\omega} \tag{8.15}$$
for all $g\in G$ and for all curves $\widetilde{C}\in H_2(\widetilde{X},\mathbb{Z})\simeq\mathbb{Z}^{19}$.

3. Divide by $|G|$.

Note that setting $\widetilde{C}=g(\widetilde{C})$ in $H_2(\widetilde{X},\mathbb{Z})$ yields by definition the coinvariant homology $H_2(\widetilde{X},\mathbb{Z})_G$, see eq. (4.19). Now, in general, this might not be enough to describe $H_2(X,\mathbb{Z})$ since there are potentially higher differentials in the Cartan-Leray spectral sequence, eq. (5.11). However, as we discovered in 5, there are no such subtleties in our case and, according to eq. (5.8), the homology classes of rational curves on X are identified with the coinvariant homology on $\widetilde{X}$.

So all we have to do is to implement the relation eq. (8.15) in the expression for the prepotential on $\widetilde{X}$, eq. (7.8). This can be done by restricting the complexified Kähler class ω, only allowing classes that yield the same result when integrated over $\widetilde{C}$ or $g(\widetilde{C})$. Those classes are precisely the G-invariant Kähler classes, see eq. (4.9). Hence, we would like to set[24]

$$\begin{aligned}\omega &= t^1_R\phi+t^2_R\tau_1+t^3_R\tau_2\\ &=(t^1_R+5t^2_R+5t^3_R)\phi\\ &\quad+t^2_R\pi_1^{-1}(5\sigma)+t^2_R\pi_1^{-1}(-2\alpha_1)+t^2_R\pi_1^{-1}(-\alpha_2)+t^2_R\pi_1^{-1}(\alpha_8)\\ &\quad+t^3_R\pi_2^{-1}(5\sigma)+t^3_R\pi_2^{-1}(-2\alpha_1)+t^3_R\pi_2^{-1}(-\alpha_2)+t^3_R\pi_2^{-1}(\alpha_8),\end{aligned} \tag{8.17}$$

where we used eqs. (4.11) and (4.8). Unfortunately, this is not yet the correct way to implement the relations in eq. (8.15). In fact, this restriction on ω is too strong. Recall that two of the relations in the coinvariant homology, see eq. (4.22), only have to hold with

---

[24] This particular choice of generators has the added advantage that its basis elements also span the G-invariant Kähler cone [57]
$$\mathcal{K}(\widetilde{X})^G=\operatorname{span}_{\mathbb{R}_>}\{\phi,\tau_1,\tau_2\}. \tag{8.16}$$
As a consequence, the Fourier series of the prepotential will only contain non-negative powers.

a certain multiplicity, namely

$$3\big(\sigma\underline{\times}\mu - \sigma\underline{\times}\sigma\big) = 0, \quad 3\big(\sigma\underline{\times}\nu - \sigma\underline{\times}\sigma\big) = 0. \tag{8.18}$$

However, demanding that ω be G-invariant enforces a stronger relation, one without the multiplicity, and, hence, kills the torsion information.

To capture the torsion information, we need to add two more Kähler classes which feel the torsion curves. We choose

$$\begin{aligned}
\beta_1 &= \pi_1^{-1}\big(-6\sigma + 3\theta_{21} + 4\theta_{31} + 2\theta_{32} + 4\theta_{41} + 2\theta_{42} + 6\mu\big) \\
&\quad + \pi_2^{-1}\big(6\sigma - 3\theta_{21} - 4\theta_{31} - 2\theta_{32} - 4\theta_{41} - 2\theta_{42} - 6\mu\big) \\
&= \pi_1^{-1}\big(-24\sigma + \alpha_1 + 3\alpha_2 + 6\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + \alpha_7 + 3\alpha_8\big) \\
&\quad + \pi_2^{-1}\big(24\sigma - \alpha_1 - 3\alpha_2 - 6\alpha_3 - 4\alpha_4 - 3\alpha_5 - 3\alpha_6 - \alpha_7 - 3\alpha_8\big) \\
&= PD\big(\sigma\underline{\times}\mu\big) - PD\big(\mu\underline{\times}\sigma\big), \\
\beta_2 &= -27\phi + \pi_1^{-1}\big(12\sigma - 6\theta_{11} - 4\theta_{31} - 8\theta_{32} - 8\theta_{41} - 4\theta_{42} - 12\nu\big) \\
&\quad + \pi_2^{-1}\big(6\sigma - 3\theta_{11} - 2\theta_{31} - 4\theta_{32} - 4\theta_{41} - 2\theta_{42} - 6\nu\big) \\
&= \pi_1^{-1}\big(24\sigma - 2\alpha_1 - 4\alpha_2 - 6\alpha_3 - 4\alpha_4 - 2\alpha_5 - 6\alpha_8\big) \\
&\quad + \pi_2^{-1}\big(12\sigma - \alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5 - 3\alpha_8\big) \\
&= PD\big(\sigma\underline{\times}\nu\big) + 2PD\big(\nu\underline{\times}\sigma\big) - 45\phi.
\end{aligned} \tag{8.19}$$

These two additional Kähler classes, β_1 and β_2, have exactly the right property: They are perpendicular to all relations in the coinvariant homology, eq. (4.22), except for the last two (reproduced in eq. (8.18)) that only need to hold with multiplicity three. That is,

$$\big(\sigma\underline{\times}\theta_{mn} - \sigma\underline{\times}\theta_{11}\big)\cdot\beta_i = 0 \quad \forall m = 1,2,3,4;\ n = 0,1,2; \tag{8.20}$$

$$\big(\theta_{mn}\underline{\times}\sigma - \theta_{11}\underline{\times}\sigma\big)\cdot\beta_i = 0 \quad \forall m = 1,2,3,4;\ n = 0,1,2; \tag{8.21}$$

$$\big(\sigma\underline{\times}f - 3\,\sigma\underline{\times}\theta_{11}\big)\cdot\beta_i = 0, \qquad \big(f\underline{\times}\sigma - 3\,\theta_{11}\underline{\times}\sigma\big)\cdot\beta_i = 0, \tag{8.22}$$

$$\big(2\,\sigma\underline{\times}\sigma - \mu\underline{\times}\sigma + \sigma\underline{\times}\mu\big)\cdot\beta_i = 0, \qquad \big(\sigma\underline{\times}\sigma + \nu\underline{\times}\sigma - 2\,\sigma\underline{\times}\nu\big)\cdot\beta_i = 0 \tag{8.23}$$

for $i = 1, 2$. Moreover, with respect to the two curve classes on $\widetilde{X}$ that push-forward to the torsion curve generators, see eq. (4.29), they form a dual basis:

$$\begin{aligned}
\big(\sigma\underline{\times}\mu - \sigma\underline{\times}\sigma\big)\cdot\beta_1 &= 1, \qquad & \big(\sigma\underline{\times}\mu - \sigma\underline{\times}\sigma\big)\cdot\beta_2 &= 0, \\
\big(\sigma\underline{\times}\nu - \sigma\underline{\times}\sigma\big)\cdot\beta_1 &= 0, \qquad & \big(\sigma\underline{\times}\nu - \sigma\underline{\times}\sigma\big)\cdot\beta_2 &= 1.
\end{aligned} \tag{8.24}$$

Hence, instead of restricting ω to the 3-dimensional invariant space eq. (8.17), we now restrict ω to lie in the 5-dimensional subspace of Kähler forms

$$\omega = t_R^1\phi + t_R^2\tau_1 + t_R^3\tau_2 + t_R^4\beta_1 + t_R^5\beta_2. \tag{8.25}$$

As usual, it is more convenient to work with the Fourier-transformed variables

$$p = e^{2\pi i t_R^1}, \quad q = e^{2\pi i t_R^2}, \quad r = e^{2\pi i t_R^3}, \quad b_1 = e^{2\pi i t_R^4}, \quad b_2 = e^{2\pi i t_R^5}, \tag{8.26}$$

where

$$b_1^3 = 1, \quad b_2^3 = 1 \tag{8.27}$$

since they correspond to the torsion curve classes. The 5-dimensional subset of the Kähler moduli space parametrized by the t_R^a can, of course, be expressed in terms of special linear combinations of the 19 Kähler moduli t^a defined in eq. (7.9). Then, using the definitions eqs. (7.10) and (8.26), we obtain the relations

$$\begin{array}{llll} & & p_0 = pq^5r^5 & \\ & q_0 = q^5 & & r_0 = r^5 \\ q_1 = q^{-2}b_1b_2 & q_2 = q^{-1}b_2^2 & r_1 = r^{-2}b_1^2b_2^2 & r_2 = r^{-1}b_2 \\ q_3 = 1 & q_4 = b_1b_2^2 & r_3 = 1 & r_4 = b_1^2b_2 \\ q_5 = b_2 & q_6 = 1 & r_5 = b_2^2 & r_6 = 1 \\ q_7 = b_1 & q_8 = q & r_7 = b_1^2 & r_8 = q. \end{array} \tag{8.28}$$

We now have everything in place to compute the genus 0 prepotential on $X = \widetilde{X}/G$. Imposing the curve relations eq. (8.15) on the instanton sum for the prepotential on $\widetilde{X}$, eq. (7.8), is completely equivalent to substituting eq. (8.28) in the final expression for the prepotential on $\widetilde{X}$, eq.(7.15). The non-perturbative prepotential on the quotient is then $\frac{1}{|G|}$ times the prepotential on the covering space after the replacement. The result is

$$\begin{aligned} \mathcal{F}^{\text{np}}_{X,0}(p,q,r,b_1,b_2) &= \frac{1}{|G|}\, \mathcal{F}^{\text{np}}_{\widetilde{X},0}(p_0,q_0,\ldots,q_8,r_0,\ldots,r_8)\Big|_{p_0=pq^5r^5,\ldots,r_8=q} \\ &= \frac{1}{9}pA(q,b_1,b_2)A(r,b_1^{-1},b_2^{-1}) + O(p^2), \end{aligned} \tag{8.29}$$

where we defined the auxiliary function, see eq. (7.16),

$$\begin{aligned} A(q,b_1,b_2) &= \widetilde{A}\big(q^5, q^{-2}b_1b_2, q^{-1}b_2^2, 1, b_1b_2^2, b_2, 1, b_1, q\big) \\ &= \Theta_{E_8}\big(q^3;\, q^2b_1^2b_2^2,\, qb_2,\, 1,\, b_1^2b_2,\, b_2^2,\, 1,\, b_1^2,\, q^{-1}\big)P\big(q^3\big)^{12} \end{aligned} \tag{8.30}$$

and an analogous expression for $A(r,b_1^{-1},b_2^{-1})$. Expanding $A(q,b_1,b_2)$ as a power series, we find

$$\begin{aligned} A(q,b_1,b_2) &= \Big(1 + 4q + 14q^2 + 28q^3 + 57q^4 + 84q^5 + 148q^6 + 196q^7 + \cdots\Big) \\ &\quad \times (1+b_1+b_1^2)(1+b_2+b_2^2)P(q^3)^{12} \\ &= \Big(1 + 4q + 14q^2 + 40q^3 + 105q^4 + 252q^5 + 574q^6 + 1240q^7 + \cdots\Big) \\ &\qquad \times (1+b_1+b_1^2)(1+b_2+b_2^2) \\ &\in \mathbb{Z}[[q]] \otimes \mathbb{Z}[b_1,b_2]\big/\big\langle b_1^3 = 1, b_2^3 = 1\big\rangle\,. \end{aligned} \tag{8.31}$$

Since the series expansion is invariant under $(b_1,b_2) \mapsto (b_1^{-1},b_2^{-1}) = (b_1^2,b_2^2)$, we only have to replace $q \mapsto r$ in eq. (8.31) to obtain the series expansion for $A(r,b_1^{-1},b_2^{-1})$.

To conclude, we have computed an explicit closed form for the prepotential on $X = \widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ at linear order in p. This was done by starting with the prepotential on $\widetilde{X}$ and suitably "modding out" the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action. One can now expand the prepotential eq. (8.29) as a power series and compare it with the general form eq. (8.13), thereby reading off the instanton numbers. The impatient reader can find them in 2 on page 44. However, before we come to that, we will calculate the prepotential on X directly in the next subsection. In the course of this alternative computation, we will find that the expression eq. (8.29) can be significantly simplified.

8.3 Directly on the quotient X

Instead of working with generic dP_9 surfaces, one can also work directly with the special surfaces in eq. (2.2a) and eq. (2.2b). In order to admit a vertical $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ group action, they have a special complex structure such that

- There are 9 sections, $MW(B_i) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.
- The elliptic fibration $B_i \to \mathbb{P}^1$ has $4I_3$ Kodaira fibers.

The three irreducible components of each of the four I_3 fibers are permuted by the four different $\mathbb{Z}_3$ subgroups of G. Therefore, the quotient $X = \widetilde{X}/G$ is still fibered by Abelian surfaces, having 4 singular fibers of the type $T^2 \times I_0$ and 4 singular fibers of the type $I_0 \times T^2$. We can immediately identify the following curves on the quotient X:

- 9 sections s_{ij} in $MW(X) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$, all distinguished by $H_2(X, \mathbb{Z})_{\text{tors}} = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.
- The fiber classes f_1 and f_2 under the two different elliptic fibrations.

Following exactly the same reasoning as in 7.1, one can write down the instanton contribution from the pseudo-sections to the genus 0 prepotential *directly on the quotient* X. The result is

$$\mathcal{F}^{\text{np}}_{X,0} = \sum_{\substack{s_{ij}\in \\ MW(X)}} e^{2\pi \mathrm{i} \int_{s_{ij}} \omega} \left(\sum_{m=0}^{\infty} p(m) e^{2\pi \mathrm{i} m \int_{f_1} \omega} \right)^4 \left(\sum_{n=0}^{\infty} p(n) e^{2\pi \mathrm{i} n \int_{f_2} \omega} \right)^4 + (\text{contribution of multi-sections}). \quad (8.32)$$

We now pick variables for the complexified Kähler moduli space on X such that

$$e^{2\pi \mathrm{i} \int_{s_{ij}} \omega} = p b_1^i b_2^j, \quad e^{2\pi \mathrm{i} \int_{f_1} \omega} = q, \quad e^{2\pi \mathrm{i} \int_{f_2} \omega} = r. \quad (8.33)$$

Expanding the prepotential in these variables, we obtain

$$\begin{aligned} \mathcal{F}^{\text{np}}_{X,0}(p,q,r,b_1,b_2) &= \left(\sum_{i,j=0}^{2} p b_1^i b_2^j \right) P(q)^4 P(r)^4 + O(p^2) \\ &= p(1+b_1+b_1^2)(1+b_2+b_2^2) P(q)^4 P(r)^4 + O(p^2). \end{aligned} \quad (8.34)$$

Note that this expression appears to be distinct from eq. (8.29). However, although the two formulas look very different, they must be identical functions of p, q, r, b_1, b_2. Indeed, as we now show, this is the case. Note that the difficult part in the first expression for the prepotential is the E_8 theta function in the function A, see eq. (8.30). First, let us ignore b_1 and b_2 for the moment, that is, set $b_1 = b_2 = 1$, and recall [58]

Theorem 3 (Zagier).

$$\Theta_{E_8}\big(q^3;\, q^2,\, q,\, 1,\, 1,\, 1,\, 1,\, 1,\, q^{-1}\big)P\big(q^3\big)^{12} = 9P(q)^4 \quad \in \mathbb{Z}[[q]]. \tag{8.35}$$

Using this identity, we can eliminate the E_8 theta function from the function $A(q,1,1)$. A short computation then shows the equality of the two expressions for the prepotential, eqns (8.34) and (8.29).

Putting b_1 and b_2 back into $A(q,b_1,b_2)$, it is very suggestive that Zagier's identity ought to be generalized to

$$\begin{aligned}\Theta_{E_8}\big(q^3;\, q^2b_1^2b_2^2,\, qb_2,\, 1,\, b_1^2b_2,\, b_2^2,\, 1,\, b_1^2,\, q^{-1}\big)P\big(q^3\big)^{12} &= \\ &= (1+b_1+b_1^2)(1+b_2+b_2^2)P(q)^4 \\ &\qquad \in \mathbb{Z}[[q]]\otimes\mathbb{Z}[b_1,b_2]\big/\big\langle b_1^3=1, b_2^3=1\big\rangle\,. \end{aligned} \tag{8.36}$$

Using a computer, we have expanded both sides of eq. (8.36) up to degree 10 and found agreement. This generalized identity implies the equality of the two expressions

$$\begin{aligned}\Big\{p\text{-linear part of eq. (8.29)}\Big\} &= \frac{1}{9}pA(q,b_1,b_2)A(r,b_1^{-1},b_2^{-1}) \\ &= \frac{1}{9}p\big(1+b_1+b_1^2\big)^2\big(1+b_2+b_2^2\big)^2P(q)^4P(r)^4 \\ &= p(1+b_1+b_1^2)(1+b_2+b_2^2)P(q)^4P(r)^4 \\ &= \Big\{p\text{-linear part of eq. (8.34)}\Big\}\end{aligned} \tag{8.37}$$

for the genus 0 prepotential at linear order in p, where we used that $b_1^3 = 1 = b_2^3$. We conclude that the two expressions for the prepotential on X in eqs. (8.34) and (8.29) are indeed the same function.

Expanding our formula for the instanton generated genus 0 prepotential as a power series and comparing it with the general form given in eq. (8.13), one can finally read off the instanton numbers computed using the A-model. We will do this in the following subsection.

8.4 Instanton numbers

Recall from eq. (5.40) that to correctly distinguish all homology classes of curves, we need 5 numbers

$$(n_1,n_2,n_3,m_1,m_2)\in\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3\simeq H_2\big(X,\mathbb{Z}\big). \tag{8.38}$$

The effect of the torsion homology classes is that, for any curve on X, we can assign quantum numbers $m_1,m_2\in\{0,1,2\}$ in addition to the degrees $n_1,n_2,n_3\in\mathbb{Z}$. With this in mind, and using eq. (8.26), the general form of the instanton expression eq. (8.13) becomes

$$\mathcal{F}^{\text{np}}_{X,0}(p,q,r,\,b_1,b_2) = \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}\\ m_1,m_2\in\mathbb{Z}_3}} n_{(n_1,n_2,n_3,m_1,m_2)}\,\mathrm{Li}_3\Big(p^{n_1}q^{n_2}r^{n_3}b_1^{m_1}b_2^{m_2}\Big). \tag{8.39}$$

n_2 \ n_3	0	1	2	3	4	5	6	7	8	9
0	1	4	14	40	105	252	574	1240	2580	5180
1	4	16	56	160	420	1008	2296	4960	10320	20720
2	14	56	196	560	1470	3528	8036	17360	36120	72520
3	40	160	560	1600	4200	10080	22960	49600	103200	207200
4	105	420	1470	4200	11025	26460	60270	130200	270900	543900
5	252	1008	3528	10080	26460	63504	144648	312480	650160	1305360
6	574	2296	8036	22960	60270	144648	329476	711760	1480920	2973320
7	1240	4960	17360	49600	130200	312480	711760	1537600	3199200	6423200
8	2580	10320	36120	103200	270900	650160	1480920	3199200	6656400	13364400
9	5180	20720	72520	207200	543900	1305360	2973320	6423200	13364400	26832400

Table 2: Instanton numbers $n_{(1,n_2,n_3,*,*)}$ computable in the A-model. In this case (for $n_1 = 1$), the instanton number is independent of the torsion part of the homology class.

where $n_{(n_1,n_2,n_3,m_1,m_2)}$ is the number of instantons in the given homology class. Comparing this with the series expansion of the formula for the prepotential, either eq. (8.29) or (8.34), allows us to read off the instanton numbers.

As we explained previously, our A-model computation only yielded the genus 0 prepotential up to linear order in p, that is, for $n_1 \leq 1$. The constant part in p vanishes, so all of these instanton numbers are zero,

$$n_{(0,n_2,n_3,m_1,m_2)} = 0 \quad \forall n_2, n_3 \in \mathbb{Z},\ m_1, m_2 \in \mathbb{Z}_3. \tag{8.40}$$

At linear order in p, that is, $n_1 = 1$, the instanton numbers do not vanish. Interestingly, the instanton number does not depend on the torsion part of the homology class. That is,

$$n_{(1,n_2,n_3,m_1,m_2)} = n_{(1,n_2,n_3,0,0)} \quad \forall m_1, m_2 \in \{0,1,2\}. \tag{8.41}$$

The underlying reason for this is another geometric $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action. Unlike $G \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$, this additional group acts on X and has fixed points, see Part B [30], section 6. On the homology classes $(1, n_2, n_3, m_1, m_2)$ its action is generated by $m_1 \mapsto (m_1 + 1)$ mod 3 and $m_2 \mapsto (m_2 + 1)$ mod 3. Since the prepotential must respect this symmetry, the corresponding instanton numbers are equal.

We list the instanton numbers for $n_2, n_3 \leq 9$ in 2. Note the symmetry under the exchange $n_2 \leftrightarrow n_3$. This is already visible in the expression for the prepotential, which is invariant under the exchange $q \leftrightarrow r$,

$$\mathcal{F}^{\text{np}}_{X,0}(p, r, q, b_1, b_2) = \left(\sum_{i,j=0}^{2} p b_1^i b_2^j \right) P(q)^4 P(r)^4 + O(p^2) = \mathcal{F}^{\text{np}}_{X,0}(p, q, r, b_1, b_2). \tag{8.42}$$

The underlying geometric reason is that we can exchange the factors in the fiber product

$$\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2 \simeq B_2 \times_{\mathbb{P}^1} B_1. \tag{8.43}$$

Unwinding the definitions, one can show that this geometric exchange corresponds precisely to the exchange of q and r.

The instanton numbers calculated using the A-model, and presented above, have one glaring limitation. Namely, they are restricted to $n_1 \leq 1$. That is, we can only compute the prepotential to linear order in p. Using mirror symmetry, we will be able to overcome this restriction in Part B [30].

8.5 The partial quotient $\overline{\mathbf{X}}$

Since $G = G_1 \times G_2 = \mathbb{Z}_3 \times \mathbb{Z}_3$ is generated by two independent $\mathbb{Z}_3$ actions, there are the obvious partial quotients

$$\begin{tikzcd}
 & \widetilde{X} \arrow[dl, "\text{mod } G_1"'] \arrow[dr, "\text{mod } G_2"] \arrow[dd, "\text{mod } G" description] & \\
\overline{X} = \widetilde{X}/G_1 \arrow[dr, "\text{mod } G_2"'] & & \widetilde{X}/G_2 \, . \arrow[dl, "\text{mod } G_1"] \\
 & X &
\end{tikzcd} \tag{8.44}$$

Having just computed the prepotential on X, there is little intrinsic interest in the simpler partial quotients. However, note that the G_1 quotient $\overline{X} = \widetilde{X}/G_1$ is again a toric variety since G_1 acts only by phase rotations on the coordinates, see eq. (2.3a). This observation will enable us to compute the instanton numbers using the B-model, as we will in Part B [30]. To this end, we will need the correct variable substitution analogous to eq. (8.28) but for the final G_2 quotient $X = \overline{X}/G_2$. This is why we will analyze the partial quotient $\overline{X}$ in this subsection. In the same way as for the full $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ quotient, we can compute part of its prepotential by properly descending $\widetilde{X} \to \overline{X}$.

Because we will have to compare our basis for divisors with the basis that is natural in toric geometry, let us first have a closer look at the G_1 invariant cohomology of $\widetilde{X}$. First, the G_1 invariant homology of the dP_9 surfaces is

$$H_2(B_i, \mathbb{Z})^{G_1} = \operatorname{span}_{\mathbb{Z}} \Big\{ f,\, t,\, u,\, v \Big\}, \tag{8.45}$$

where f and t are the $G_1 \times G_2$ invariant divisors, see eq. (4.7) and[25]

$$\begin{aligned} u &= \theta_{21} + \theta_{31} + \theta_{41} + 3\mu = 6f + 6\sigma - 2\alpha_1 - \alpha_2 \\ v &= 2t + \theta_{11} = -3\alpha_1 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 3\alpha_8 \end{aligned} \tag{8.46}$$

are only G_1 but not G_2-invariant. As in 4.2, pulling these back yields a basis for the G_1-invariant divisor classes of the Calabi-Yau threefold. We define

$$\upsilon_1 = \pi_1^{-1}(u), \quad \upsilon_2 = \pi_2^{-1}(u), \quad \psi_1 = \pi_1^{-1}(v), \quad \psi_2 = \pi_2^{-1}(v) \tag{8.47}$$

---

[25] At this point it is not obvious why we choose $2t + \theta_{11}$ instead of just θ_{11} for the final generator of the G_1-invariant cohomology. As we will see below, this particular basis choice is better adapted to the Kähler cone.

in addition to eq. (4.10). As usual, we will not distinguish between divisors and their duals in cohomology, see 12. With this abuse of notation, we obtain the basis

$$H^2\big(\widetilde{X},\mathbb{Z}\big)^{G_1} = \operatorname{span}_{\mathbb{Z}}\Big\{\phi,\, \tau_1,\, \upsilon_1,\, \psi_1,\, \tau_2,\, \upsilon_2,\, \psi_2\Big\}. \tag{8.48}$$

All products between these cohomology classes are determined by the relations

$$\begin{aligned}H^{\mathrm{ev}}\big(\widetilde{X},\mathbb{Q}\big)^{G_1} = \mathbb{Q}[\phi,\tau_1,\upsilon_1,\psi_1,\tau_2,\upsilon_2,\psi_2]\Big/\big\langle &\phi^2,\, \tau_1\phi = 3\tau_1^2,\, \tau_2\phi = 3\tau_2^2,\\ &\phi\upsilon_1 = 3\tau_1^2,\, \phi\upsilon_2 = 3\tau_2^2,\, \phi\psi_1 = 6\tau_1^2,\, \phi\psi_2 = 6\tau_2^2,\, \tau_1\upsilon_1 = 3\tau_1^2,\, \tau_2\upsilon_2 = 3\tau_2^2,\\ &\tau_1\psi_1 = 3\tau_1^2,\, \tau_2\psi_2 = 3\tau_2^2,\, \upsilon_1\upsilon_1 = 3\tau_1^2,\, \upsilon_2\upsilon_2 = 3\tau_2^2,\, \upsilon_1\psi_1 = 6\tau_1^2,\, \upsilon_2\psi_2 = 6\tau_2^2,\\ &\psi_1\psi_1 = 6\tau_1^2,\, \psi_2\psi_2 = 6\tau_2^2,\, (\tau_1-\upsilon_1)(\tau_2-\upsilon_2),\, (2\upsilon_1-\psi_1)(2\upsilon_2-\psi_2),\\ &(2\upsilon_1-\psi_1)(2\tau_2-\psi_2),\, (2\upsilon_2-\psi_2)(\tau_1-\upsilon_1)\big\rangle.\end{aligned} \tag{8.49}$$

Using the above relations, we find that any triple intersection can be rewritten as a multiple of $\tau_1^2\tau_2 = 3\{\mathrm{pt.}\}$. Therefore, the non-vanishing intersection numbers are

$$\begin{array}{ccccc} \phi\tau_1\tau_2 = 9 & \phi\tau_1\upsilon_2 = 9 & \phi\tau_1\psi_2 = 18 & \phi\upsilon_1\tau_2 = 9 & \phi\upsilon_1\upsilon_2 = 9 \\ \phi\upsilon_1\psi_2 = 18 & \phi\psi_1\tau_2 = 18 & \phi\psi_1\upsilon_2 = 18 & \phi\psi_1\psi_2 = 36 & \tau_1^2\tau_2 = 3 \\ \tau_1^2\upsilon_2 = 3 & \tau_1^2\psi_2 = 6 & \tau_1\upsilon_1\tau_2 = 9 & \tau_1\upsilon_1\upsilon_2 = 9 & \tau_1\upsilon_1\psi_2 = 18 \\ \tau_1\psi_1\tau_2 = 9 & \tau_1\psi_1\upsilon_2 = 9 & \tau_1\psi_1\psi_2 = 18 & \tau_1\tau_2^2 = 3 & \tau_1\tau_2\upsilon_2 = 9 \\ \tau_1\tau_2\psi_2 = 9 & \tau_1\upsilon_2^2 = 9 & \tau_1\upsilon_2\psi_2 = 18 & \tau_1\psi_2^2 = 18 & \upsilon_1^2\tau_2 = 9 \\ \upsilon_1^2\upsilon_2 = 9 & \upsilon_1^2\psi_2 = 18 & \upsilon_1\psi_1\tau_2 = 18 & \upsilon_1\psi_1\upsilon_2 = 18 & \upsilon_1\psi_1\psi_2 = 36 \\ \upsilon_1\tau_2^2 = 3 & \upsilon_1\tau_2\upsilon_2 = 9 & \upsilon_1\tau_2\psi_2 = 9 & \upsilon_1\upsilon_2^2 = 9 & \upsilon_1\upsilon_2\psi_2 = 18 \\ \upsilon_1\psi_2^2 = 18 & \psi_1^2\tau_2 = 18 & \psi_1^2\upsilon_2 = 18 & \psi_1^2\psi_2 = 36 & \psi_1\tau_2^2 = 6 \\ \psi_1\tau_2\upsilon_2 = 18 & \psi_1\tau_2\psi_2 = 18 & \psi_1\upsilon_2^2 = 18 & \psi_1\upsilon_2\psi_2 = 36 & \psi_1\psi_2^2 = 36. \end{array} \tag{8.50}$$

The G_1-invariant Kähler cone on B_i consists of the potential Kähler classes in $H^2(B_i,\mathbb{Z})^{G_1}$. It can be computed [57] as the dual of the cone of effective curves on B_i. The effective curves are [59]

Theorem 4 (Looijenga). *The cone of effective curves on a dP_9 surface B is generated by the following curve classes $e \in H_2(B_i,\mathbb{Z})$:*

1. *The exceptional curves ($e^2 = -1$). These are the elements of the Mordell-Weil group $MW(B)$.*
2. *The irreducible components of singular Kodaira fibers ($e^2 = -2$).*
3. *The "future cone" of the positive classes ($e^2 \geq 1$).*

For the $\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetric dP_9 surfaces B_1, B_2 that we are interested in, the Mordell-Weil group consists of the 9 elements given in eq. (3.4). Furthermore, the $4I_3$ Kodaira fibers have 12 irreducible components $\theta_{10}, \dots, \theta_{42}$. The positive classes do not yield any extra constraints on the dual cone. The Kähler cone

$$\mathcal{K}(B_i)^{G_1} = \operatorname{span}_{\mathbb{R}_>}\Big\{\kappa_1,\, \kappa_2,\, \kappa_3,\, \kappa_4,\, \kappa_5,\, \kappa_6,\, \kappa_7,\, \kappa_8\Big\} \quad \subset H^2\big(B_i,\mathbb{Z}\big)^{G_1} \tag{8.51}$$

turns out to be non-simplicial with edges

$$\begin{aligned} &\kappa_1 = f \qquad \kappa_2 = t && \kappa_3 = u \qquad \kappa_4 = v \\ &\kappa_5 = 3t + f - v && \kappa_6 = 3t + u - v \\ &\kappa_7 = f - u + v && \kappa_8 = 3t + f - u. \end{aligned} \tag{8.52}$$

For future reference we note that the intersection matrix of the Kähler cone generators on B_i is

$(-)\cdot(-)$	κ_1	κ_2	κ_3	κ_4	κ_5	κ_6	κ_7	κ_8
κ_1	0	3	3	6	3	6	3	6
κ_2	3	1	3	3	3	3	3	3
κ_3	3	3	3	6	6	6	6	9
κ_4	6	3	6	6	9	9	6	9
κ_5	3	3	6	9	3	6	6	6
κ_6	6	3	6	9	6	6	9	9
κ_7	3	3	6	6	6	9	3	6
κ_8	6	3	9	9	6	9	6	6

(8.53)

We note that G_1 and G_2 commute. Hence, G_2 acts on the G_1-invariant homology and Kähler cone. Using the explicit group action, see eq. (3.17), one finds

$$g_2 \begin{pmatrix} f \\ t \\ u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 0 & -1 \\ 0 & 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} f \\ t \\ u \\ v \end{pmatrix} \tag{8.54}$$

and

$$\Big(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6, \kappa_7, \kappa_8\Big) \overset{g_2}{\mapsto} \Big(\kappa_1, \kappa_2, \kappa_5, \kappa_6, \kappa_7, \kappa_8, \kappa_3, \kappa_4\Big) \tag{8.55}$$

Using the Kähler cone on the base dP_9 surfaces, the Kähler cone on $\overline{X}$ is finally found [57] to be

$$\begin{aligned} \mathcal{K}\big(\overline{X}\big) &= \mathcal{K}\big(\widetilde{X}\big)^{G_1} \\ &= \operatorname{span}_{\mathbb{R}_>} \Big\{\phi,\, \tau_1,\, \pi_1^*(\kappa_3), \dots, \pi_1^*(\kappa_8),\; \tau_2,\, \pi_2^*(\kappa_3), \dots, \pi_2^*(\kappa_8)\Big\}. \end{aligned} \tag{8.56}$$

Let us now return to the instanton counting on $\overline{X} = \widetilde{X}/G_1$. Recall from eq. (5.16) that

$$H_2\big(\overline{X}, \mathbb{Z}\big) = \mathbb{Z}^7 \oplus \mathbb{Z}_3. \tag{8.57}$$

Using the same trick as in 8.2, we can determine the prepotential on $\overline{X}$. We pick restricted Kähler moduli

$$\omega = t_R^1 \phi + t_R^2 \tau_1 + t_R^3 \upsilon_1 + t_R^4 \psi_1 + t_R^5 \tau_2 + t_R^6 \upsilon_2 + t_R^7 \psi_2 + t_R^8 \beta_1 \tag{8.58}$$

corresponding to a basis[26] for the G_1-invariant cohomology, see eq. (8.48), and one additional generator β_1 which detects the generator of

$$H_2\big(\widetilde{X},\mathbb{Z}\big)_{G_1,\mathrm{tors}} = \mathbb{Z}_3 = H_2\big(\overline{X},\mathbb{Z}\big)_{\mathrm{tors}}, \tag{8.59}$$

see eq. (8.19). The Fourier transformed variables, which we will use in the following, are

$$\begin{aligned} &\qquad\qquad P = e^{2\pi i t^1_R}, \\ Q_1 &= e^{2\pi i t^2_R}, \quad Q_2 = e^{2\pi i t^3_R}, \quad Q_3 = e^{2\pi i t^4_R}, \\ R_1 &= e^{2\pi i t^5_R}, \quad R_2 = e^{2\pi i t^6_R}, \quad R_3 = e^{2\pi i t^7_R},, \end{aligned} \tag{8.60}$$

and

$$b_1 = e^{2\pi i t^8_R}, \tag{8.61}$$

where

$$b_1^3 = 1. \tag{8.62}$$

The relations between the restricted variables and the full 19 variables are

$$\begin{array}{llll} & p_0 = PQ_1^5Q_2^6R_1^5R_2^6 & & \\ & q_0 = Q_1^5Q_2^6 & & r_0 = R_1^5R_2^6 \\ q_1 = Q_1^{-2}Q_2^{-2}Q_3^{-3}b_1 & q_2 = Q_1^{-1}Q_2^{-1} & r_1 = R_1^{-2}R_2^{-2}R_3^{-3}b_1^2 & r_2 = R_1^{-1}R_2^{-1} \\ q_3 = Q_3^3 & q_4 = Q_3^2b_1 & r_3 = R_3^3 & r_4 = R_3^2b_1^2 \\ q_5 = Q_3 & q_6 = 1 & r_5 = R_3 & r_6 = 1 \\ q_7 = b_1 & q_8 = Q_1Q_3^3 & r_7 = b_1^2 & r_8 = R_1R_3^3. \end{array} \tag{8.63}$$

As done previously for the full quotient, we now substitute these variables into the formula for the prepotential on the covering space $\widetilde{X}$, see eq. (7.15), and divide by $|G_1| = 3$. The result is

$$\begin{aligned} \mathcal{F}^{\mathrm{np}}_{\overline{X},0}(P,Q_1,Q_2,Q_3,R_1,R_2,R_3,b_1) &= \frac{1}{|G_1|}\,\mathcal{F}^{\mathrm{np}}_{\widetilde{X},0}(p,q_0,\ldots,q_8,p_0,\ldots,p_8) \qquad (8.64)\\ &= \frac{1}{3}P\,\overline{A}(Q_1,Q_2,Q_3,b_1)\overline{A}(R_1,R_2,R_3,b_1^{-1}) + O(P^2), \end{aligned}$$

where

$$\begin{aligned} \overline{A}(Q_1,Q_2,Q_3,b_1) = \Theta_{E_8}\Big(&Q_1^3Q_2^3Q_3^6;\, Q_1^2Q_2^2Q_3^3b_1^2,\, Q_1Q_2,\, Q_3^{-3}, \\ &Q_3^{-2}b_1^2,\, Q_3^{-1},\, 1,\, b_1^2,\, Q_1^{-1}Q_3^{-3}\Big)\,P\big(Q_1^3Q_2^3Q_3^6\big)^{12} \end{aligned} \tag{8.65}$$

---

[26] Note that the 7 generators ϕ, τ_1, υ_1, ψ_1, τ_2, υ_2, ψ_2 are the edges of one maximal simplicial subcone of the Kähler cone. This ensures again that the Fourier series of the prepotential will only contain positive powers.

and the analogous expression for $\overline{A}(R_1, R_2, R_3, b_1^{-1})$. Expanding $\overline{A}(Q_1, Q_2, Q_3, b_1)$ as a power series, we find

$$\begin{aligned}
\overline{A}(Q_1, Q_2, Q_3, b_1) = (1 + b_1 + b_1^2)\times & \\
\times\big(1+ & \, Q_2 + Q_2Q_3 + Q_1Q_2Q_3 + 3Q_1Q_2Q_3^2 + 3Q_1Q_2^2Q_3^2 + Q_1Q_2Q_3^3+ \\
& + Q_1^2Q_2Q_3^3 + 3Q_1Q_2^2Q_3^3 + 3Q_1^2Q_2^2Q_3^3 + Q_1^2Q_2Q_3^4 + Q_1Q_2^3Q_3^3+ \\
& + Q_1^2Q_2^3Q_3^3 + 9Q_1^2Q_2^2Q_3^4 + 9Q_1^2Q_2^3Q_3^4 + 3Q_1^2Q_2^2Q_3^5+ \\
& + 3Q_1^3Q_2^2Q_3^5 + Q_1^2Q_2^4Q_3^4 + 9Q_1^2Q_2^3Q_3^5+ \\
& + 9Q_1^3Q_2^3Q_3^5 + 3Q_1^3Q_2^2Q_3^6 + 3Q_1^2Q_2^4Q_3^5 + Q_1^2Q_2^3Q_3^6+ \\
& + 3Q_1^3Q_2^4Q_3^5 + 25Q_1^3Q_2^3Q_3^6 + Q_1^2Q_2^4Q_3^6+ \\
& + (\text{total degree} \geq 13)\big) \qquad \in \mathbb{Z}[[Q_1, Q_2, Q_3]] \otimes \mathbb{Z}[b_1] / \big\langle b_1^3 = 1 \big\rangle
\end{aligned} \tag{8.66}$$

Finally, we note that we can now compute the prepotential on $X = \widetilde{X}/G$ in terms of the prepotential on $\overline{X} = \widetilde{X}/G_1$. One can easily show that the correct substitution of variables is

$$\begin{gathered}
P = p \\
\begin{aligned}
Q_1 &= q & \qquad R_1 &= r \\
Q_2 &= b_2 & \qquad R_2 &= b_2^2 \\
Q_3 &= b_2 & \qquad R_3 &= b_2^2.
\end{aligned}
\end{gathered} \tag{8.67}$$

Obviously one obtains exactly the same as prepotential as in eq. (8.29), where we divided out $G = G_1 \times G_2$ in one step rather than first G_1 and then G_2. However, as we will show in the companion paper Part B, one can use toric mirror symmetry to compute any desired term in the prepotential on $\overline{X}$. Knowing the above substitution, eq. (8.67), will enable us to find the prepotential on $X = \overline{X}/G_2$ beyond linear order in p, including its b_2 torsion expansion.

9. Conclusion

The goal of this paper is to investigate rational curves on the Calabi-Yau threefold X, which is the $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ quotient of its universal cover $\widetilde{X}$. Its Hodge numbers and integral homology are

$$h^{p,q}\big(X\big) = \begin{matrix}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 3 & & 3 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{matrix}\,, \quad H_i\big(X, \mathbb{Z}\big) \simeq \begin{cases}
\mathbb{Z} & i = 6 \\
0 & i = 5 \\
\mathbb{Z}^3 \oplus \big(\mathbb{Z}_3\big)^2 & i = 4 \\
\mathbb{Z}^8 \oplus \big(\mathbb{Z}_3\big)^2 & i = 3 \\
\mathbb{Z}^3 \oplus \big(\mathbb{Z}_3\big)^2 & i = 2 \\
\big(\mathbb{Z}_3\big)^2 & i = 1 \\
\mathbb{Z} & i = 0.
\end{cases} \tag{9.1}$$

Interestingly, this is one of the few known examples of Calabi-Yau manifolds whose degree-2 homology has a finite part (*torsion*). The prepotential is a function of the 3 free generators p, q, r and the 2 torsion generators b_1, b_2. We found a closed formula for the genus zero prepotential

$$\mathcal{F}^{\mathrm{np}}_{X,0}(p,q,r,b_1,b_2) = \Big(\sum_{i,j=0}^{2} p b_1^i b_2^j\Big) P(q)^4 P(r)^4 + O(p^2) = \sum_{i,j=0}^{2} \operatorname{Li}_3(p b_1^i b_2^j) + \cdots \tag{9.2}$$

to linear order in p. This allows us to derive part of the instanton numbers on X, distinguishing the torsion part of the curve class in the integral homology. The corresponding instantons are listed in 2 on page 44.

Clearly, we would like to obtain the complete prepotential and not just up to linear order in p. However, this is very difficult to do directly. In Part B [30], we will use mirror symmetry to attack this problem. There, we will find a way to obtain the higher order terms as well. The final result, limited only by computing power, will be

$$\begin{aligned}
&\mathcal{F}^{\mathrm{np}}_{X,0}(p,q,r,b_1,b_2) = \mathcal{F}^{\mathrm{np}}_{X^*,0}(p,q,r,b_1,b_2) \\
&\quad = \sum_{i,j=0}^{2} \Big(\operatorname{Li}_3(p b_1^i b_2^j) + 4\operatorname{Li}_3(pq b_1^i b_2^j) + 4\operatorname{Li}_3(pr b_1^i b_2^j) \\
&\qquad + 14\operatorname{Li}_3(pq^2 b_1^i b_2^j) + 16\operatorname{Li}_3(pqr b_1^i b_2^j) + 14\operatorname{Li}_3(pr^2 b_1^i b_2^j) \\
&\qquad + 40\operatorname{Li}_3(pq^3 b_1^i b_2^j) + 56\operatorname{Li}_3(pq^2 r b_1^i b_2^j) + 56\operatorname{Li}_3(pqr^2 b_1^i b_2^j) \\
&\qquad + 40\operatorname{Li}_3(pr^3 b_1^i b_2^j) + 105\operatorname{Li}_3(pq^4 b_1^i b_2^j) + 160\operatorname{Li}_3(pq^3 r b_1^i b_2^j) \\
&\qquad + 196\operatorname{Li}_3(pq^2 r^2 b_1^i b_2^j) + 160\operatorname{Li}_3(pqr^3 b_1^i b_2^j) + 105\operatorname{Li}_3(pr^4 b_1^i b_2^j) \\
&\qquad - 2\operatorname{Li}_3(p^2 q b_1^i b_2^j) - 2\operatorname{Li}_3(p^2 r b_1^i b_2^j) - 28\operatorname{Li}_3(p^2 q^2 b_1^i b_2^j) \\
&\qquad + 32\operatorname{Li}_3(p^2 qr b_1^i b_2^j) - 28\operatorname{Li}_3(p^2 r^2 b_1^i b_2^j) - 192\operatorname{Li}_3(p^2 q^3 b_1^i b_2^j) \\
&\qquad + 440\operatorname{Li}_3(p^2 q^2 r b_1^i b_2^j) + 440\operatorname{Li}_3(p^2 q r^2 b_1^i b_2^j) - 192\operatorname{Li}_3(p^2 r^3 b_1^i b_2^j) \Big) \\
&\quad + 3\operatorname{Li}_3(p^3 q) + 3\operatorname{Li}_3(p^3 r) \\
&\quad + 9\operatorname{Li}_3(p^3 q^2) + 27 \sum_{(i,j)\neq(0,0)} \operatorname{Li}_3(p^3 q^2 b_1^i b_2^j) \\
&\quad + 9\operatorname{Li}_3(p^3 r^2) + 27 \sum_{(i,j)\neq(0,0)} \operatorname{Li}_3(p^3 r^2 b_1^i b_2^j) \\
&\quad + 27\operatorname{Li}_3(p^3 qr) + 81 \sum_{(i,j)\neq(0,0)} \operatorname{Li}_3(p^3 qr b_1^i b_2^j) \\
&\quad + \big(\text{total } p,q,r\text{-degree } \geq 6\big).
\end{aligned} \tag{9.3}$$

This provides some interesting examples of instanton numbers that do depend on the torsion part of their homology class, see 3.

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$n_{(3,n_2,n_3,0,0)}$

n_2 \ n_3	0	1	2
0	0	**3**	**36**
1	**3**	**108**	
2	**36**		

$n_{(3,n_2,n_3,m_1,m_2)}$, $(m_1,m_2) \neq (0,0)$

n_2 \ n_3	0	1	2
0	0	**0**	**27**
1	**0**	**81**	
2	**27**		

Table 3: Some of the instanton numbers $n_{(n_1,n_2,n_3,m_1,m_2)}$ computed by mirror symmetry. The entries marked in **bold** depend non-trivially on the torsion part of their respective homology class.

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A. Duology

A.1 Poincaré duality and equalities

For any closed, connected, oriented d-dimensional manifold Y there are non-singular[27] pairings

$$\begin{aligned} H_k(Y,\mathbb{Z})_{\text{free}} \times H^k(Y,\mathbb{Z})_{\text{free}} &\to \mathbb{Z}, & (S,\varphi) &\mapsto \int_S \varphi, \\ H^k(Y,\mathbb{Z})_{\text{free}} \times H^{d-k}(Y,\mathbb{Z})_{\text{free}} &\to \mathbb{Z}, & (\varphi,\psi) &\mapsto \int_Y \varphi\wedge\psi, \\ H_k(Y,\mathbb{Z})_{\text{free}} \times H_{d-k}(Y,\mathbb{Z})_{\text{free}} &\to \mathbb{Z}, & (M,N) &\mapsto M\cdot N. \end{aligned} \tag{A.1}$$

The consequence is that the corresponding (co)homology groups are of the same rank. Moreover, if a group G acts orientation-preservingly on Y then the corresponding (co)homology groups are dual G-representations.

However, the "best" version of Poincaré duality identifies homology and cohomology including torsion, and is a map

$$PD:\ H^k(Y,\mathbb{Z}) \xrightarrow{\sim} H_{d-k}(Y,\mathbb{Z}), \quad \varphi \mapsto [Y]\cap\varphi. \tag{A.2}$$

This map PD is an isomorphism; by abuse of notation we will denote the inverse by PD as well. In full generality, the map PD is the cap-product with the fundamental class.

---

[27] A bilinear map is non-singular if, when written in terms of integral bases, it is represented by a square matrix of determinant 1.

Ignoring torsion, we can also describe PD on the level of differential forms as follows: Consider a $(d-k)$-dimensional submanifold $S \subset Y$. Then the k-form $PD(S)$ is the Thom class of the normal bundle $N_{Y|S}$, that is, a bump k-form along the normal directions of S. Note that PD does not involve any duality. If there is an orientation-preserving G-action on Y, then $H^k(Y,\mathbb{Z}) \simeq H_{d-k}(Y,\mathbb{Z})$ are isomorphic group representations.

A.2 Tate duality

Looking at the result for $\mathbb{Z}_3 \times \mathbb{Z}_3$ group (co)homology in eq. (4.34), there seems to be the following relation

$$H_i\big(G, R^\vee\big)_{\text{tors}} \simeq H^{i+1}\big(G, R\big)_{\text{tors}} \tag{A.3}$$

between group homology and group cohomology. In fact, this is a general property known as Tate duality. Recall that the Tate cohomology groups unify group homology and cohomology into

$$\widehat{H}^i(G,M) = \begin{cases} H^i(G,M) & i > 0 \\ M^G/(\text{tr})M & i = 0 \\ \ker(\text{tr})/IM & i = -1 \\ H_{-i-1}(G,M) & i < -1, \end{cases} \tag{A.4}$$

where M is any G-module. If M is $\mathbb{Z}$-torsion free, that is, a representation of G on a lattice $\mathbb{Z}^n$, then [61]

$$\widehat{H}^i\big(G, \text{Hom}(M,\mathbb{Z})\big) \simeq \text{Hom}\left[\widehat{H}^{-i}(G,M), \mathbb{Q}/\mathbb{Z}\right] \tag{A.5}$$

In particular, setting $M = R$ proves eq. (A.3).

B. Relations amongst divisors

In 3.1, eq. (3.7) we chose one particular basis for the homology of the dP_9 surfaces, namely

$$H_2(B_i,\mathbb{Z}) = \text{span}_{\mathbb{Z}} \Big\{\sigma,\ f,\ \theta_{11},\ \theta_{21},\ \theta_{31},\ \theta_{32},\ \theta_{41},\ \theta_{42},\ \mu,\ \nu\Big\}. \tag{B.1}$$

In this appendix we give the expansion of the other curves of interest in terms of this chosen basis. The expansion of any other curve can be found using its intersection numbers with the 10 base curves.

The 9 sections forming the Mordell-Weil group intersect the vertical divisors according

JHEP10(2007)022

to eq. (3.16), and they do not intersect amongst themselves. Hence,

$$\begin{aligned}
\sigma &= \sigma, \\
\mu &= \mu, \\
\mu \boxplus \mu &= -\sigma - f + \theta_{21} + \theta_{31} + \theta_{41} + 2\mu, \\
\nu &= \nu, \\
\nu \boxplus \mu &= -\sigma - f + \theta_{31} + \theta_{32} + \theta_{41} + \mu + \nu, \\
\nu \boxplus \mu \boxplus \mu &= -2\sigma - 2f + \theta_{21} + \theta_{31} + \theta_{32} + 2\theta_{41} + \theta_{42} + 2\mu + \nu, \\
\nu \boxplus \nu &= -\sigma - f + \theta_{11} + \theta_{32} + \theta_{41} + 2\nu, \\
\nu \boxplus \nu \boxplus \mu &= -2\sigma - 2f + \theta_{11} + \theta_{31} + \theta_{32} + 2\theta_{41} + \theta_{42} + \mu + 2\nu, \\
\nu \boxplus \nu \boxplus \mu \boxplus \mu &= -3\sigma - 3f + \theta_{11} + \theta_{21} + 2\theta_{31} + 2\theta_{32} + 2\theta_{41} + \theta_{42} + 2\mu + 2\nu.
\end{aligned} \tag{B.2}$$

Finally, the components of $i = 1, \ldots, 4$ distinct I_3 Kodaira fibers intersect as

$(-) \cdot (-)$	θ_{i0}	θ_{i1}	θ_{i2}
θ_{i0}	-2	1	1
θ_{i1}	1	-2	1
θ_{i2}	1	1	$-2.$

(B.3)

This lets us express the two components θ_{12}, θ_{22} that are not part of our chosen basis as

$$\begin{aligned}
\theta_{12} &= 3\sigma + 3f - 2\theta_{11} - \theta_{31} - 2\theta_{32} - 2\theta_{41} - \theta_{42} - 3\nu, \\
\theta_{22} &= 3\sigma + 3f - 2\theta_{21} - 2\theta_{31} - \theta_{32} - 2\theta_{41} - \theta_{42} - 3\mu.
\end{aligned} \tag{B.4}$$

C. Image of group homology

The purpose of this appendix is to find the image

$$\mathbb{Z}_3 \simeq H_3\Big(G_{12}; \mathbb{Z}\Big) \longrightarrow H_3\Big(G; \mathbb{Z}\Big) \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3\,. \tag{C.1}$$

The obvious way to get an explicit handle on this map is to extend the inclusion $\mathbb{Z}G_{12} \subset \mathbb{Z}G$ to a chain map of the corresponding resolutions of $\mathbb{Z}$. Applying $- \otimes \mathbb{Z}$ to the resolution then makes the image of the homology group clear.

To write down the resolution, define the following **t**race and **d**ifference maps in the group ring:

$$t_1 = \sum_{i=0}^{2} \big(g_1\big)^i\,, \quad t_2 = \sum_{i=0}^{2} \big(g_2\big)^i\,, \quad d_1 = 1 - g_1\,, \quad d_2 = 1 - g_2\,. \tag{C.2}$$

Using these, we write down the following chain map between the resolutions. From that, one can easily determine the pushforward of the homology groups as

$$\begin{tikzcd}
\mathbb{Z}G_{12} \arrow[r, "\sum(g_1g_2)^i"] \arrow[d, "(\cdots)"] & \mathbb{Z}G_{12} \arrow[r, "1-g_1g_2"] \arrow[d, "\begin{pmatrix} 1 & g_1 & g_1^2 & 1 \end{pmatrix}"] & \mathbb{Z}G_{12} \arrow[r, "\sum(g_1g_2)^i"] \arrow[d, "\begin{pmatrix} 1 & 1+g_1+g_1g_2 & g_1^2 \end{pmatrix}"] & \mathbb{Z}G_{12} \arrow[r, "1-g_1g_2"] \arrow[d, "\begin{pmatrix} g_2 & 1 \end{pmatrix}"] & \mathbb{Z}G_{12} \arrow[d, hook] \\
\oplus_5\mathbb{Z}G \arrow[r, "\begin{pmatrix} t_1 & 0 & 0 & 0 \\ -d_2 & d_1 & 0 & 0 \\ 0 & t_2 & t_1 & 0 \\ 0 & 0 & -d_2 & d_1 \\ 0 & 0 & 0 & t_2 \end{pmatrix}"'] & \oplus_4\mathbb{Z}G \arrow[r, "\begin{pmatrix} d_1 & 0 & 0 \\ d_2 & t_1 & 0 \\ 0 & -d_2 & d_1 \\ 0 & 0 & h_2 \end{pmatrix}"'] & \oplus_3\mathbb{Z}G \arrow[r, "\begin{pmatrix} t_1 & 0 \\ -d_2 & d_1 \\ 0 & t_2 \end{pmatrix}"'] & \oplus_2\mathbb{Z}G \arrow[r, "\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}"'] & \oplus_1\mathbb{Z}G
\end{tikzcd}$$

$$\Downarrow \quad \text{Apply } \left(-\otimes_{\mathbb{Z}G_{12}}\mathbb{Z}\right) \text{ resp. } (-\otimes_{\mathbb{Z}G}\mathbb{Z})$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "3"] \arrow[d, "(\cdots)"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}"] & \mathbb{Z} \arrow[r, "3"] \arrow[d, "\begin{pmatrix} 1 & 3 & 1 \end{pmatrix}"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] & \mathbb{Z} \arrow[d, equal] \\
\mathbb{Z}^5 \arrow[r, "\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}" description] & \mathbb{Z}^4 \arrow[r, "\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}" description] & \mathbb{Z}^3 \arrow[r, "\begin{pmatrix} 3 & 0 \\ 0 & 0 \\ 0 & 3 \end{pmatrix}" description] & \mathbb{Z}^2 \arrow[r, "\begin{pmatrix} 0 \\ 0 \end{pmatrix}" description] & \mathbb{Z}
\end{tikzcd} \tag{C.3}$$

$$\Downarrow \quad \text{Homology}$$

$$\begin{tikzcd}
0 \arrow[d, equal] & \mathbb{Z}_3 \arrow[d, equal] & 0 \arrow[d, equal] & \mathbb{Z}_3 \arrow[d, equal] & \mathbb{Z} \arrow[d, equal] \\
H_4\big(G_{12};\mathbb{Z}\big) \arrow[d] & H_3\big(G_{12};\mathbb{Z}\big) \arrow[d, "\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}"] & H_2\big(G_{12};\mathbb{Z}\big) \arrow[d] & H_1\big(G_{12};\mathbb{Z}\big) \arrow[d, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] & H_0\big(G_{12};\mathbb{Z}\big) \arrow[d, equal] \\
H_4\big(G;\mathbb{Z}\big) \arrow[d, equal] & H_3\big(G;\mathbb{Z}\big) \arrow[d, equal] & H_2\big(G;\mathbb{Z}\big) \arrow[d, equal] & H_1\big(G;\mathbb{Z}\big) \arrow[d, equal] & H_0\big(G;\mathbb{Z}\big) \arrow[d, equal] \\
\big(\mathbb{Z}_3\big)^2 & \big(\mathbb{Z}_3\big)^3 & \mathbb{Z}_3 & \big(\mathbb{Z}_3\big)^2 & \mathbb{Z}\,.
\end{tikzcd}$$

It is much easier to determine the image under the inclusion $G_1 \subset G$ and $G_2 \subset G$. Using the same bases as in eq. (C.3), they are

$$\begin{aligned} H_3\big(G_1;\mathbb{Z}\big) &= \mathbb{Z}_3 \xrightarrow{(1\;0\;0)} \big(\mathbb{Z}_3\big)^3 = H_3\big(G;\mathbb{Z}\big) \\ H_3\big(G_2;\mathbb{Z}\big) &= \mathbb{Z}_3 \xrightarrow{(0\;0\;1)} \big(\mathbb{Z}_3\big)^3 = H_3\big(G;\mathbb{Z}\big)\,. \end{aligned} \tag{C.4}$$

References

[1] P. Candelas, X.C. De La Ossa, P.S. Green and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, *Nucl. Phys.* **B 359** (1991) 21.

[2] P.S. Aspinwall, D.R. Morrison and M. Gross, *Stable singularities in string theory*, *Commun. Math. Phys.* **178** (1996) 115 [`hep-th/9503208`].

[3] V. Batyrev and M. Kreuzer, *Integral cohomology and mirror symmetry for Calabi-Yau 3-folds*, `math.AG/0505432`.

[4] M. Gross and S. Pavanelli, *A Calabi-Yau threefold with Brauer group* $(Z/8Z)^2$, `math.AG/0512182`.

[5] S. Ferrara, J.A. Harvey, A. Strominger and C. Vafa, *Second quantized mirror symmetry*, *Phys. Lett.* **B 361** (1995) 59 [`hep-th/9505162`].

[6] P.S. Aspinwall, *An* $N = 2$ *dual pair and a phase transition*, *Nucl. Phys.* **B 460** (1996) 57 [`hep-th/9510142`].

[7] V. Braun, Y.-H. He and B.A. Ovrut, *Yukawa couplings in heterotic standard models*, *JHEP* **04** (2006) 019 [`hep-th/0601204`].

[8] V. Braun, Y.-H. He and B.A. Ovrut, *Stability of the minimal heterotic standard model bundle*, *JHEP* **06** (2006) 032 [`hep-th/0602073`].

[9] V. Braun and B.A. Ovrut, *Stabilizing moduli with a positive cosmological constant in heterotic M-theory*, *JHEP* **07** (2006) 035 [`hep-th/0603088`].

[10] R.P. Thomas, *An obstructed bundle on a Calabi-Yau* 3*-fold*, *Adv. Theor. Math. Phys.* **3** (1999) 567.

[11] E. Lima, B.A. Ovrut and J. Park, *Five-brane superpotentials in heterotic M-theory*, *Nucl. Phys.* **B 626** (2002) 113 [`hep-th/0102046`].

[12] E. Lima, B.A. Ovrut, J. Park and R. Reinbacher, *Non-perturbative superpotential from membrane instantons in heterotic M-theory*, *Nucl. Phys.* **B 614** (2001) 117 [`hep-th/0101049`].

[13] E.I. Buchbinder, R. Donagi and B.A. Ovrut, *Superpotentials for vector bundle moduli*, *Nucl. Phys.* **B 653** (2003) 400 [`hep-th/0205190`].

[14] E.I. Buchbinder, R. Donagi and B.A. Ovrut, *Vector bundle moduli superpotentials in heterotic superstrings and M-theory*, *JHEP* **07** (2002) 066 [`hep-th/0206203`].

[15] E. Buchbinder and B.A. Ovrut, *Vector bundle moduli*, *Russ. Phys. J.* **45** (2002) 662.

[16] E.I. Buchbinder and B.A. Ovrut, *Vacuum stability in heterotic M-theory*, *Phys. Rev.* **D 69** (2004) 086010 [`hep-th/0310112`].

[17] E. Buchbinder, R. Donagi and B.A. Ovrut, *Vector bundle moduli and small instanton transitions*, *JHEP* **06** (2002) 054 [`hep-th/0202084`].

[18] J. Distler, *Resurrecting* $(2, 0)$ *compactifications*, *Phys. Lett.* **B 188** (1987) 431.

[19] J. Distler and B.R. Greene, *Aspects of* $(2, 0)$ *string compactifications*, *Nucl. Phys.* **B 304** (1988) 1.

[20] E. Silverstein and E. Witten, *Criteria for conformal invariance of* $(0, 2)$ *models*, *Nucl. Phys.* **B 444** (1995) 161 [`hep-th/9503212`].

[21] A. Basu and S. Sethi, *World-sheet stability of* $(0, 2)$ *linear* σ*-models*, *Phys. Rev.* **D 68** (2003) 025003 [`hep-th/0303066`].

[22] C. Beasley and E. Witten, *Residues and world-sheet instantons*, *JHEP* **10** (2003) 065 [`hep-th/0304115`].

[23] V. Braun, Y.-H. He, B.A. Ovrut and T. Pantev, *The exact MSSM spectrum from string theory*, *JHEP* **05** (2006) 043 [`hep-th/0512177`].

[24] V. Braun, Y.-H. He, B.A. Ovrut and T. Pantev, *A heterotic standard model*, *Phys. Lett.* **B 618** (2005) 252 [`hep-th/0501070`].

[25] V. Braun, Y.-H. He, B.A. Ovrut and T. Pantev, *A standard model from the $E_8 \times E_8$ heterotic superstring*, *JHEP* **06** (2005) 039 [`hep-th/0502155`].

[26] V. Braun, Y.-H. He, B.A. Ovrut and T. Pantev, *Vector bundle extensions, sheaf cohomology and the heterotic standard model*, *Adv. Theor. Math. Phys.* **10** (2006) 4 [`hep-th/0505041`].

[27] C.F. Doran and J.W. Morgan, *Algebraic topology of Calabi-Yau threefolds in toric varieties*, `math.AG/0605074`.

[28] S. Gukov, S. Kachru, X. Liu and L. McAllister, *Heterotic moduli stabilization with fractional Chern-Simons invariants*, *Phys. Rev.* **D 69** (2004) 086008 [`hep-th/0310159`].

[29] S. Hosono, M. Saito and J. Stienstra, *On the mirror symmetry conjecture for Schoen's Calabi-Yau 3-folds*, `alg-geom/9709027`. Given at Taniguchi Symposium on Integrable Systems and Algebraic Geometry, Kyoto, Japan, 7-11 Jul (1997).

[30] V. Braun, M. Kreuzer, B.A. Ovrut and E. Scheidegger, *Worldsheet instantons and torsion curves. Part B: mirror symmetry*, to appear.

[31] M. Kreuzer and H. Skarke, *PALP: a package for analyzing lattice polytopes with applications to toric geometry*, *Comput. Phys. Commun.* **157** (2004) 87 [`math.NA/0204356`].

[32] V. Braun, M. Kreuzer, B.A. Ovrut and E. Scheidegger, *Worldsheet instantons, torsion curves and non-perturbative superpotentials*, *Phys. Lett.* **B 649** (2007) 334 [`hep-th/0703134`].

[33] C. Schoen, *On fiber products of rational elliptic surfaces with section*, *Math. Z.* **197** (1988) 177.

[34] D. Lüst, S. Reffert, E. Scheidegger and S. Stieberger, *Resolved toroidal orbifolds and their orientifolds*, `hep-th/0609014`.

[35] V. Braun, B.A. Ovrut, T. Pantev and R. Reinbacher, *Elliptic Calabi-Yau threefolds with $Z_3 \times Z_3$ Wilson lines*, *JHEP* **12** (2004) 062 [`hep-th/0410055`].

[36] B.A. Ovrut, T. Pantev and R. Reinbacher, *Torus-fibered Calabi-Yau threefolds with non-trivial fundamental group*, *JHEP* **05** (2003) 040 [`hep-th/0212221`].

[37] V. Braun, *K-theory torsion*, `hep-th/0005103`.

[38] S. Hosono, M.H. Saito and A. Takahashi, *Holomorphic anomaly equation and BPS state counting of rational elliptic surface*, *Adv. Theor. Math. Phys.* **3** (1999) 177 [`hep-th/9901151`].

[39] A. Klemm, P. Mayr and C. Vafa, *BPS states of exceptional non-critical strings*, `hep-th/9607139`.

[40] D. Maulik and R. Pandharipande, *New calculations in Gromov-Witten theory*, `math.AG/0601395`.

[41] U. Persson, *Configurations of Kodaira fibers on rational elliptic surfaces*, *Math. Z.* **205** (1990) 1.

[42] K. Oguiso and T. Shioda, *The Mordell-Weil lattice of a rational elliptic surface*, *Comment. Math. Univ. St. Paul.* **40** (1991) 83.

[43] T. Shioda, *On the Mordell-Weil lattices*, *Comment. Math. Univ. St. Paul.* **39** (1990) 211.

[44] R. Donagi, B.A. Ovrut, T. Pantev and R. Reinbacher, SU(4) *instantons on Calabi-Yau threefolds with $Z_2 \times Z_2$ fundamental group*, *JHEP* **01** (2004) 022 [`hep-th/0307273`].

[45] B.A. Ovrut, T. Pantev and R. Reinbacher, *Invariant homology on standard model manifolds*, *JHEP* **01** (2004) 059 [`hep-th/0303020`].

[46] G. Curio and D. Lüst, *A class of $N = 1$ dual string pairs and its modular superpotential*, *Int. J. Mod. Phys.* **A 12** (1997) 5847 [`hep-th/9703007`].

[47] K.S. Brown, *Cohomology of groups*, in *Graduate Texts in Mathematics* **87**, Springer-Verlag, New York (1994) (reprint of 1982 version).

[48] A. Adem and R.J. Milgram, *Cohomology of finite groups*, 2^{nd} ed., in *Grundlehren der Mathematischen Wissenschaften* **309**, Springer-Verlag, Berlin (2004).

[49] M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, *Proc. London Math. Soc.* **25** (1972) 75.

[50] J. Leray, *L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue*, *J. Math. Pures Appl.* **29** (1950) 1.

[51] J.P. Serre, *How to write mathematics*, online at http://video.google.com/videoplay?docid=8266735963828183844.

[52] H. Cartan, *Sur la cohomologie des espaces où opère un groupe: étude d'un anneau différentiel où opère un groupe*, *C. R. Acad. Sci. Paris* **226** (1948) 303.

[53] S. Eilenberg and S. MacLane, *Relations between homology and homotopy groups of spaces*, *Ann. Math.* **46** (1945) 480.

[54] P.S. Aspinwall and D.R. Morrison, *Chiral rings do not suffice: $N = (2, 2)$ theories with nonzero fundamental group*, *Phys. Lett.* **B 334** (1994) 79 [`hep-th/9406032`].

[55] R. Donagi, A. Grassi and E. Witten, *A non-perturbative superpotential with $E(8)$ symmetry*, *Mod. Phys. Lett.* **A 11** (1996) 2199 [`hep-th/9607091`].

[56] L. Göttsche and R. Pandharipande, *The quantum cohomology of blow-ups of $\mathbf{P}^2$ and enumerative geometry*, *J. Diff. Geom.* **48** (1998) 61.

[57] T.L. Gomez, S. Lukic and I. Sols, *Constraining the Kähler moduli in the heterotic standard model*, `hep-th/0512205`.

[58] D. Zagier, *A modular identity arising from mirror symmetry*, in *Integrable systems and algebraic geometry*, Kobe/Kyoto (1997), World Sci. Publ., River Edge, NJ, U.S.A. (1998).

[59] E. Looijenga, *Rational surfaces with an anticanonical cycle*, *Ann. Math.* **114** (1981) 267.

[60] G.M. Greuel, G. Pfister and H. Schönemann, *Singular* 3.0*, a computer algebra system for polynomial computations*, Centre for Computer Algebra, University of Kaiserslautern, (2005); online at http://www.singular.uni-kl.de.

[61] S. Lang, *Topics in cohomology of groups*,in *Lecture Notes in Mathematics*, **1625**, Springer-Verlag, Berlin (1996) (translated from the 1967 French original by the author).